Section 6.2: Basic theory of systems of n first order linear equations

$$\mathbf{x}' = P(t)\mathbf{x} + g(t)$$

where

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \quad P(t) = \begin{pmatrix} p_{11}(t) & \cdots & p_{1n}(t) \\ \vdots & \ddots & \vdots \\ p_{n1}(t) & \cdots & p_{nn}(t) \end{pmatrix}, \quad g(t) = \begin{pmatrix} g_1(t) \\ \vdots \\ g_n(t) \end{pmatrix},$$

Main topics:

- Existence and unicity of solutions
- Principle of superposition (homogenous systems, i.e. $g(t) = 0$)
- Independence of solutions and the Wronskian (homogeneous systems)
- General solutions (homogeneous systems)

Two generalizations with respect to Section 3.3:

- n linear DE
- $P(t)$ not necessarily constant in t.

Consider the system of n linear first-order DEs:

$$\mathbf{x}' = P(t)\mathbf{x} + g(t)$$

where

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \quad P(t) = \begin{pmatrix} p_{11}(t) & \cdots & p_{1n}(t) \\ \vdots & \ddots & \vdots \\ p_{n1}(t) & \cdots & p_{nn}(t) \end{pmatrix}, \quad g(t) = \begin{pmatrix} g_1(t) \\ \vdots \\ g_n(t) \end{pmatrix},$$

Theorem (Theorem 6.2.1)

*If $P(t)$ and $g(t)$ are continuous matrix functions on a open interval I and $t_0 \in I$, then the IVP: $\mathbf{x}' = P(t)\mathbf{x} + g(t)$ with initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$ has a **unique solution** $\mathbf{x} = \mathbf{x}(t)$ with t in I.*

Definition

We say that $\mathbf{x}$ is a **linear combination** of $\mathbf{x_1}, \ldots, \mathbf{x_k}$, written $\mathbf{x} = c_1\mathbf{x_1} + \cdots + c_k\mathbf{x_k}$, if there are constants $c_1, \ldots, c_k$ such that $\mathbf{x}(t) = c_1\mathbf{x_1}(t) + \cdots + c_k\mathbf{x_k}(t)$ for all t.

Theorem (Theorem 6.2.2, Principle of superposition)

*If $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_k$ are k solutions of the **homogeneous** linear system $\mathbf{x}' = P(t)\mathbf{x}$ on an open interval I, then any linear combination $c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_k\mathbf{x}_k$, where the c_j's are constants, is a solution of the system in I.*

Let I be an open interval and for $t \in I$ set:

$$\mathbf{x_1}(t) = \begin{pmatrix} x_{11}(t) \\ \vdots \\ x_{n1}(t) \end{pmatrix}, \ldots, \mathbf{x_n}(t) = \begin{pmatrix} x_{1n}(t) \\ \vdots \\ x_{nn}(t) \end{pmatrix}.$$

Let

$$\mathbf{X} = [\mathbf{x_1}, \ldots, \mathbf{x_n}] = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix}$$

be the $n \times n$ matrix-valued function on I having $\mathbf{x_1}, \ldots, \mathbf{x_n}$ as column vectors.

Definition

The **Wronskian** of $\mathbf{x_1}, \ldots, \mathbf{x_n}$ is the determinant

$$W[\mathbf{x_1}, \ldots \mathbf{x_n}] = \det(\mathbf{X})$$

Remarks:

- $\mathbf{X} : t \mapsto \mathbf{X}(t)$ is a matrix-valued function of $t \in I$.
- $W[\mathbf{x_1}, \ldots \mathbf{x_n}]$ is a function of $t \in I$: for every fixed $t \in I$, its value $W[\mathbf{x_1}, \ldots \mathbf{x_n}](t)$ is the determinant of the matrix $\mathbf{X}(t)$.

Definition

The vector-valued functions $\mathbf{x_1} = \begin{pmatrix} x_{11} \\ \vdots \\ x_{n1} \end{pmatrix}, \ldots, \mathbf{x_n} = \begin{pmatrix} x_{1n} \\ \vdots \\ x_{nn} \end{pmatrix}$ defined on the interval I are said **linearly dependent** in I if there are constants $c_1, \ldots, c_n$ (not all zero and *independent of* $t \in I$) such that

$$c_1\mathbf{x_1}(t) + \cdots + c_n\mathbf{x_n}(t) = 0 \text{ for all } t \in I.$$

If $\mathbf{x_1}, \ldots, \mathbf{x_n}$ are not linearly dependent, we say they are **linearly independent**. This means that $c_1 = \cdots = c_n = 0$ are the only constants such that

$$c_1\mathbf{x_1}(t) + \cdots + c_n\mathbf{x_n}(t) = 0 \text{ for all } t \in I.$$

Let $\mathbf{x_1}, \ldots, \mathbf{x_n}$ are n solutions of the textbfhomogeneous linear system $\mathbf{x}' = P(t)\mathbf{x}$ of n linear DE on the open interval I.

Theorem (Theorem 6.2.5)

Suppose that $P(t)$ is a continuous function of $t \in I$:

- *If $\mathbf{x_1}, \ldots, \mathbf{x_n}$ are linearly independent on I, then $W[\mathbf{x_1}, \ldots \mathbf{x_n}](t) \neq 0$ for all $t \in I$.*
- *If $\mathbf{x_1}, \ldots, \mathbf{x_n}$ are linearly dependent on I, then $W[\mathbf{x_1}, \ldots \mathbf{x_n}](t) = 0$ for all $t \in I$.*

Consequence: if there is $t_0 \in I$ such that $W[\mathbf{x_1}, \ldots \mathbf{x_n}](t_0) \neq 0$, then $\mathbf{x_1}, \ldots, \mathbf{x_n}$ are linearly independent on I.

Definition

Let $\mathbf{x_1}, \ldots, \mathbf{x_n}$ be a set of n solutions of the homogeneous linear system $\mathbf{x}' = P(t)\mathbf{x}$ of n linear DEs. If $\mathbf{x_1}, \ldots, \mathbf{x_n}$ are linearly independent on I, then we say that they form a **fundamental system of solutions** on I.

Theorem (Theorem 6.2.6)

Let $\mathbf{x_1}, \ldots, \mathbf{x_n}$ be a fundamental system of solutions on I for the homogeneous linear system $\mathbf{x}' = P(t)\mathbf{x}$. Then any solution of the this system on I is of the form $\mathbf{x} = c_1\mathbf{x_1} + \cdots + c_n\mathbf{x_n}$ for some constants $c_1, \ldots, c_n$.
*This is the **general solution** of the system.*

Moreover: an intitial condition $\mathbf{x}(t_0) = \mathbf{x}_0$, where $\mathbf{x}_0 = \begin{pmatrix} x_{10} \\ \vdots \\ x_{n0} \end{pmatrix}$ is a constant vector, uniquely determines the constants $c_1, \cdots, c_n$. The solution to the IVP is unique.

Conclusion:

- The general solution of a **homogenous** system of n linear first order DEs (with $P(t)$ continuous) is a linear combination of n **linearly independent** solutions **(same n)**
- To find the general solution it is enough to find n linearly independent solutions.
- To check if n solutions $\mathbf{x_1}, \ldots, \mathbf{x_n}$ are linearly independent, we compute their Wronskian.

Example:

Consider the system $\mathbf{x}' = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 0 & 2 \\ -1 & -2 & 4 \end{pmatrix} \mathbf{x}$ on the interval $I = \mathbb{R}$.

One can verify that the functions $\mathbf{x}_1(t) = e^t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{x}_2(t) = e^{2t} \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ and $\mathbf{x}_3(t) = e^{3t} \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$ are solutions of this system.

- Computing their Wronskian, determine whether $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$ form a fundamental set of solutions for this system of DE on $\mathbb{R}$.
- Write down the general solution.
- Determine the unique solution satisfying the initial condition $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$