SECTION 6.2

EXAMPLE $\boldsymbol{x}' = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 0 & 2 \\ -1 & -2 & 4 \end{pmatrix} \boldsymbol{x}$ on $I = \mathbb{R}$

One can verify that $\boldsymbol{x}_1(t) = e^{t}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\boldsymbol{x}_2(t) = e^{2t}\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$, $\boldsymbol{x}_3(t) = e^{3t}\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$ are solutions of this system.

- Computing their Wronskian, determine if $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3$ form a fundamental system of solutions

This is an autonomous homogeneous system

$P(t) = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 0 & 2 \\ -1 & -2 & 4 \end{pmatrix}$ is constant in t, in particular continuous

Gauss' method to compute det

$\begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

det = product diag entries

It is enough to check that for some $t_0 \in \mathbb{R}$

$$W[\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3](t_0) \neq 0$$

Choose $t_0 = 0$. Then $W[\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3](0) = \begin{vmatrix} 1 & 2 & 1 \\ 1 & 0 & -1 \\ 1 & 1 & -1 \end{vmatrix} = 2 \neq 0$

Thus $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3$ form a fundamental system of solutions.

- Write down the general solution of this system:

$$\boldsymbol{x}(t) = c_1 \boldsymbol{x}_1(t) + c_2 \boldsymbol{x}_2(t) + c_3 \boldsymbol{x}_3(t)$$
$$= c_1 e^{t}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2 e^{2t}\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + c_3 e^{3t}\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

where $c_1, c_2, c_3 \in \mathbb{R}$ and $t \in I = \mathbb{R}$

- Determine the unique solution such that $\boldsymbol{x}(0) = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

We solve the system $\boldsymbol{x}(0) = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ for c_1, c_2, c_3, i.e.

$c_1\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + c_3\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ and find $\begin{cases} c_1 = 9/2 \\ c_2 = -3 \\ c_3 = 5/2 \end{cases}$

The unique solution of the IVP is:

$$\boldsymbol{x}(t) = \frac{9}{2} e^{t}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - 3 e^{2t}\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + \frac{5}{2} e^{3t}\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$