Section 6.3: Homogenous linear systems with constant coeffs.

Consider the system $\mathbf{x}' = A\mathbf{x}$ where A is a $n \times n$ matrix with real coefficients.

Theorem (Theorem 6.3.1)

Suppose that:

(1) *A has **real** (not necessarily distinct) eigenvalues $\lambda_1, \cdots, \lambda_n$,*

(2) *A has eigenvectors $\mathbf{v}_1, \cdots, \mathbf{v}_n$ associated with the eigenvalues $\lambda_1, \cdots, \lambda_n$, respectively, so that $\mathbf{v}_1, \cdots, \mathbf{v}_n$ are **linearly independent**.*

Then the vector functions

$$\mathbf{x}_1(t) = e^{\lambda_1 t}\mathbf{v}_1, \ldots, \mathbf{x}_n(t) = e^{\lambda_n t}\mathbf{v}_n$$

form a fundamental set of solutions of the system $\mathbf{x}' = A\mathbf{x}$ on $\mathbb{R} = (-\infty, +\infty)$.

*The **general solution** of $\mathbf{x}' = A\mathbf{x}$ on $\mathbb{R}$ is*

$$\mathbf{x}(t) = C_1 e^{\lambda_1 t}\mathbf{v}_1 + \cdots + C_n e^{\lambda_n t}\mathbf{v}_n$$

where $t \in \mathbb{R}$ and $C_1, \ldots, C_n$ are constants.

- (2) is satisfied if the eigenvalues $\lambda_1, \ldots, \lambda_n$ are all distinct (Corollary 6.3.2).
- both (1) and (2) are satisfied if A is **symmetric**, i.e. $A = A^T$, where A^T denotes the transpose of A.

Example:

Find the general solution of the system of linear differential equations $\mathbf{x}' = A\mathbf{x}$ where

$$A = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}$$

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Theorem 6.3.1 applies because $A = A^T$.

Characteristic equation: $\det(A - \lambda I) = 0$, i.e. $-(\lambda - 8)(\lambda + 1)^2 = 0$.

Eigenvalues: $\lambda_1 = 8$, $\lambda_2 = -1$ (double root: we say that $\lambda_2 = -1$ has algebraic multiplicity 2).

Eigenvectors of eigenvalue $\lambda_1 = 8$ are $\mathbf{v} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} C$, where $C \neq 0$.

Eigenvectors of eigenvalue $\lambda_2 = -1$ are $\mathbf{v} = \begin{pmatrix} c_1 \\ -2(c_1 + c_2) \\ c_2 \end{pmatrix}$, where c_1, c_2 not both zero.

General solution: $\mathbf{x}(t) = C_1 e^{8t} \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} + C_2 e^{-t} \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + C_3 e^{-t} \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$ where $C_1, C_2, C_3 \in \mathbb{R}$.

Definition

We say that an $n \times n$ real matrix A is **nondefective** if there is a set of n linearly independent vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ which are eigenvectors of A. Otherwise, we say that A is **defective**.

Using this definition, we can restate the assumptions (1) and (2) in Theorem 6.3.1 as follows:

> Suppose that A is an $n \times n$ nondefective matrix with real eigenvalues $\lambda_1, \ldots, \lambda_n$. Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be n linealry independent corresponding eigenvectors (they exist as A is nondefective).
> Then etc.

We also restate the following in terms of nondefective matrices:

- If A has n distinct eigenvalues, then A the corresponging eigenvectors are n linearly independent vectors. So A is nondefective.
- Every symmetric matrix A (i.e. $A^T = A$) is nondefective (and moreover, it has all real eigenvalues).

In the next section we will look at the general solution of the system $\mathbf{x}'A = \mathbf{x}$ where A is nondefective but its eigenvalues are not necessarily real.