Section 6.4: Nondefective matrices with complex eigenvalues

Consider the system $\mathbf{x}' = A\mathbf{x}$ where A is a $n \times n$ with real coefficients and nondefective.

Suppose that not all eigenvalues of A are real.

Since A has real coefficients, as in the case of 2×2-matrices, we have:

- for any complex eigenvalue $\lambda = \mu + i\nu$ of A, also $\overline{\lambda}$ is an eigenvalue of A.
- if $\mathbf{v} = \mathbf{a} + i\mathbf{b}$ is eigenvector of A of eigenvalue λ, then
  - $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$ is a complex-valued solution of $\mathbf{x}' = A\mathbf{x}$.
  - the real-valued vector functions

$$\mathbf{u}(t) = \mathrm{Re}\,\mathbf{x}(t) = e^{\mu t}(\mathbf{a}\cos(\nu t) - \mathbf{b}\sin(\nu t))$$
$$\mathbf{w}(t) = \mathrm{Im}\,\mathbf{x}(t) = e^{\mu t}(\mathbf{a}\sin(\nu t) + \mathbf{b}\cos(\nu t))$$

    are linearly independent solutions of $\mathbf{x}' = A\mathbf{x}$.

Theorem

Let A be an $n \times n$ nondefective matrix. Suppose that A has:

- *complex (not real) eigenvalues* $\lambda_1, \overline{\lambda_1}, \ldots, \lambda_m, \overline{\lambda_m}$
- *real eigenvalues* $\lambda_{2m+1}, \ldots, \lambda_n$

Let

$$\mathbf{v}_1, \overline{\mathbf{v}_1}, \ldots, \mathbf{v}_m, \overline{\mathbf{v}_m} \qquad \text{and} \qquad \mathbf{v}_{2m+1}, \ldots, \mathbf{v}_n$$

be the n corresponding linearly independent eigenvectors (they exist as A is nondefective).

Then the general solution of the system $\mathbf{x}' = A\mathbf{x}$ is

$$\mathbf{x}(t) = C_1\mathbf{u}_1(t) + C_2\mathbf{w}_1(t) + \cdots + C_{2m-1}\mathbf{u}_m(t) + C_{2m}\mathbf{w}_m(t) + C_{2m+1}\mathbf{x}_{2m+1}(t) + \cdots + C_n\mathbf{x}_n(t)\,,$$

where

- *for $j = 1, \ldots, m$:* $\quad \mathbf{u}_j = \operatorname{Re}\mathbf{x}_j \quad$ *and* $\quad \mathbf{w}_j = \operatorname{Im}\mathbf{x}_j \quad$ *are the real-valued solutions corresponding to the complex eigenvalue λ_j and its complex eigenvector $\mathbf{v}_j$*
- *for $j = 2m+1, \ldots, n$:* $\quad \mathbf{x}_j \quad$ *is the solution corresponding to the real eigenvalue λ_j and its real eigenvector $\mathbf{v}_j$.*

Example: Find the general solution of $\mathbf{x}' = A\mathbf{x}$ where $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$

(cf. example in Section 3.4)

Characteristic equation: $\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & -\lambda & 1 \\ 0 & -1 & -\lambda \end{vmatrix} = 0$ i.e. $(1-\lambda)(\lambda^2+1) = 0$

Eigenvalues $\lambda_1 = i$, $\lambda_2 = -i$, $\lambda_3 = 1$.
Eigenvectors

for $\lambda_1 = i$: $\mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix} = \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}_{\mathbf{a}} + i \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{\mathbf{b}}$ and for $\lambda_1 = 1$: $\mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

Two linearly independent real solutions for $\lambda_1 = \mu + i\nu$ with $\mu = 0$, $\nu = 1$:

$$\mathbf{u}(t) = \mathrm{Re}\,\mathbf{x}(t) = e^{\mu t}(\mathbf{a}\cos(\nu t) - \mathbf{b}\sin(\nu t)) = \cos t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - \sin t \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \cos t \\ -\sin t \end{pmatrix}$$

$$\mathbf{w}(t) = \mathrm{Im}\,\mathbf{x}(t) = e^{\mu t}(\mathbf{a}\sin(\nu t) + \mathbf{b}\cos(\nu t)) = \sin t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \cos t \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \sin t \\ \cos t \end{pmatrix}$$

Solution corresponding to the real eigenvalue $\lambda_3 = 1$: $\mathbf{x}_3(t) = e^t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

General solution: $\mathbf{x}(t) = C_1\mathbf{u}(t) + C_2\mathbf{w}(t) + C_3\mathbf{x}_3(t)$, where C_1, C_2, C_3 are constant, $t \in \mathbb{R}$.