Section 7.3: Competing species

Main Topics:

- **Logistic equations,**
- **Mathematical models of competitive situtations:**
  two populations competing for scarse resources.

  Examples: two species competing for food supplies, competitions in a same economic market
- **Long time behavior of solutions near the critical solutions**

  Method: linear approximations.

A logistic model for competing species

Let $x(t)$ and $y(t)$ denote the size at time t of two interacting or competing populations for a common food or resource.

We will consider the following modelization (based on the logistic equation):

$$\frac{dx}{dt} = x(\epsilon_1 - \sigma_1 x - \alpha_1 y)$$
$$\frac{dy}{dt} = y(\epsilon_2 - \sigma_2 y - \alpha_2 x)$$

where

- ϵ_1 and ϵ_2 are the **growth rates** of the two populations,
- ϵ_1/σ_1 and ϵ_2/σ_2 are their **saturation levels**,
- α_1 measures the **interference** of the species y with the species x,
- α_2 measures the **interference** of the species x with the species y.

Example:

- We want to study the qualitative behavior of the solutions of the competing system

$$\frac{dx}{dt} = x(1 - x - y)$$
$$\frac{dy}{dt} = y(0.75 - y - 0.5x)$$

- The critical points of the system are:

$$(0,0),\ (0,0.75),\ (1,0) \text{ and } (0.5,0.5)$$

Critical points and direction field for the system

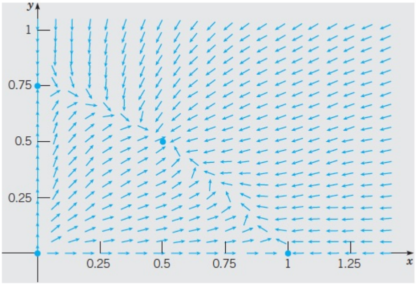

FIGURE 7.3.1 Critical points and direction field for the system (3).

For the above system, the functions

$$F(x,y) = x(1 - x - y) \;\text{ and }\; G(x,y) = y(0.75 - y - 0.5x)$$

have continuous partial derivatives of any order (in particular, up to order 2).

The system is almost linear in the neighborhood of any of the critical points.

The Jacobian matrix of the system is

$$\mathbf{J} = \begin{pmatrix} 1 - 2x - y & -x \\ -0.5y & 0.75 - 2y - 0.5x \end{pmatrix}$$

The linear approximation of the system at the critical point (X, Y) is

$$\frac{d}{dt}\begin{pmatrix} u \\ w \end{pmatrix} = \mathbf{J}(X, Y)\begin{pmatrix} u \\ w \end{pmatrix}$$

where $u = x - X$ and $w = y - Y$.

Case of $(0,0)$**:**
This solution corresponds to the extinction of both species.

The approximating linear system is

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0.75 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

The eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 0.75$.

The critical point $(0,0)$ is an unstable node of both the linear and the nonlinear systems.

The general (vector) solution of the linear system is

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{0.75t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

If $c_2 \neq 0$:

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_2 e^{-0.75t}\left[\frac{c_1}{c_2} e^{1.75t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] \sim c_2 e^{-0.75t}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad (t \to -\infty)$$

So: in the neighborhood of the origin all trajectories are tangent to the y-axis, except for one trajectory that lies on the x-axis.

Case of $(0, 0.75)$**:** In this case, the species x is extinct and y survives.

The approximate linear system near this critical point is:

$$\frac{d}{dt}\begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} 0.25 & 0 \\ -0.375 & -0.75 \end{pmatrix}\begin{pmatrix} u \\ w \end{pmatrix}$$

where $u = x$ and $w = y - 3/4$.

The eigenvalues are $\lambda_1 = 0.25$ and $\lambda_2 = -0.75$.

The critical point $(0, 0.75)$ is a saddle point and therefore it is unstable for both systems.

The general (vector) solution of the linear system is

$$\begin{pmatrix} u \\ w \end{pmatrix} = c_1 e^{0.25t}\begin{pmatrix} 8 \\ -3 \end{pmatrix} + c_2 e^{-0.75t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

If $c_2 = 0$, then one pair of trajectories approaches the critical point along the y-axis.

If $c_1 = 0$, then one pair of trajectories departs tangent to the line with slope $-3/8$.

Case of $(1, 0)$**:** In this case, the species x survives and the species y does not.
It is similar to the previous case.
The approximating linear system is

$$\frac{d}{dt}\begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 0 & 0.25 \end{pmatrix}\begin{pmatrix} u \\ w \end{pmatrix}$$

where $u = x - 1$ and $w = y$.
The eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 0.25$.
The critical point $(1, 0)$ is a saddle point and it is unstable for both systems.
The general (vector) solution of the linear system is

$$\begin{pmatrix} u \\ w \end{pmatrix} = c_1 e^{-t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{0.25t}\begin{pmatrix} 4 \\ -5 \end{pmatrix}$$

One pair of trajectories approaches the critical point along the x-axis, another leaves the critical point tangent to the line of slope $-5/4$.

Case of $(0.5, 0.5)$**:** This critical point corresponds to the coexistence of both species.
The approximating linear system is

$$\frac{d}{dt}\begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} -0.5 & -0.5 \\ -0.25 & -0.5 \end{pmatrix}\begin{pmatrix} u \\ w \end{pmatrix}$$

where $u = x - 0.5$ and $w = y - 0.5$.
The eigenvalues are $\lambda_1 = \frac{-2+\sqrt{2}}{4}$ and $\lambda_2 = \frac{-2-\sqrt{2}}{4}$.
The critical point $(0.5, 0.5)$ is an asymptotically stable node for both systems.
The general (vector) solution of the linear system is:

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{(\frac{-2+\sqrt{2}}{4})t}\begin{pmatrix} \sqrt{2} \\ -1 \end{pmatrix} + c_2 e^{(\frac{-2-\sqrt{2}}{4})t}\begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix}$$

All trajectories approach the critical point as $t \to +\infty$. Some approach the critical point along the line with slope $\sqrt{2}/2$, others along the line with slope $-\sqrt{2}/2$.

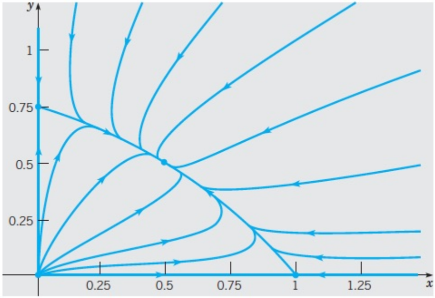

FIGURE 7.3.2 A phase portrait of the system (3).