Section 7.4: Predator-prey equations

Main Topics:

- **Mathematical models of predator-prey situations:**
  One species (the predators) lives on the the other species (the prey), the preys live on a different source of food.

  Examples:
  - Foxes live on the rabbits they prey, while the rabbits live on vegetables.
  - Lions hunt zebras, while zebras eat grass.
- **Lokta-Volterra equations,**
- **Long time behavior of solutions.**

The Lotka-Volterra equations

Let $x(t)$ denote the size of the prey and $y(t)$ the size of the predators at time t .

We consider a model for the interaction predator-prey satisfying the following assumptions:

- in the absence of predators, the prey grows at a rate proportional to the current population:
$$\frac{dx}{dt} = ax \quad \text{if } y = 0$$
- in the absence of preys, the predator dies out:
$$\frac{dy}{dt} = -cy, \quad \text{where } c > 0, \text{ if } x = 0$$
- The encounter between the two species promotes the growth of the predators and causes a shrinking of the prey. This means:

  the growth of x is affected by a term $-\alpha xy$, where $\alpha > 0$,

  the growth of y is affected by a term γxy, where $\gamma > 0$.

The equations in the resulting models are known as the **Lotka-Volterra equations**:

$$\frac{dx}{dt} = x(a - \alpha y)$$
$$\frac{dy}{dt} = y(-c + \gamma x)$$

The constants a, b, α and γ are all positive.

- a is the **growth rate** of the prey,
- c is the **death rate** of the predator,
- α and γ measure the **interactions** between the two species.

Example:

$$\frac{dx}{dt} = x(1 - 0.5y)$$
$$\frac{dy}{dt} = y(-0.75 + 0.25x)$$

for x and y positive.

Critical points: $(0, 0)$ and $(3, 2)$
The system is almost linear near each critial point.

Case of $(0, 0)$**:** Extinction of both predators and prey.
The approximating linear system is

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -0.75 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Eigenvalues of the matrix of coefficients:

- $\lambda_1 = 1$ for the eigenvector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$
- $\lambda_2 = -0.75$ for the eigenvector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

The general (vector) solution of the linear approximation is

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{-0.75t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

One pair of trajectories approaches the critical point along y-axis, another departs along x-axis. All other trajectories differenti from those along the y-axis depart from the origin.

The critical point $(0, 0)$ is an unstable saddle point for both linear and nonlinear systems.

Case of $(3,2)$**:** Survival of both predators and prey.

The approximating linear system is

$$\frac{d}{dt}\begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} 0 & -1.5 \\ 0.5 & 0 \end{pmatrix}\begin{pmatrix} u \\ w \end{pmatrix}$$

where $u = x - 3$ and $w = y - 2$ and the matrix is the value at $(3,2)$ of the Jacobian the system

$$\mathbf{J} = \begin{pmatrix} 1 - 0.5y & -0.5x \\ 0.25y & -0.75 + 0.25x \end{pmatrix}$$

The eigenvalues of the matrix $\begin{pmatrix} 0 & -1.5 \\ 0.5 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -3/2 \\ 1/2 & 0 \end{pmatrix}$ are purely imaginary:

$$\lambda_1 = i\sqrt{3}/2 \quad \text{and} \quad \lambda_2 = -i\sqrt{3}/2\,.$$

The critical point $(3,2)$ is a stable center for the above linear differential system.

What about the nonlinear system?

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What about the nonlinear system?

The table of Section 7.2 does not provide an answer in this case.

Idea (still for the linear system): find a relation between u and w.
We know that:

$$\frac{du}{dt} = -\frac{3}{2}w \quad \text{and} \quad \frac{dw}{dt} = \frac{1}{2}u$$

that is

$$\frac{dw}{du} = \frac{dw/dt}{du/dt} = \frac{\frac{1}{2}u}{-\frac{3}{2}w} = -\frac{1}{3}\frac{u}{w}$$

i.e.

$$3w\,\frac{dw}{du} = -u$$

By integration

$$u^2 + 3w^2 = C\,, \qquad C\text{=constant}$$

Concentric ellipses with center $(0,0)$ in the (u,v) plane,

i.e. concentric ellipses with center $(3,2)$ in the (x,y) plane.

Try the same method for the nonlinear system:
We know that:

$$\frac{dx}{dt} = x(1 - 0.5y)$$
$$\frac{dy}{dt} = y(-0.75 + 0.25x)$$

So

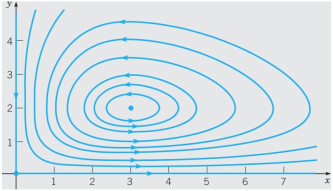
$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-0.75 + 0.25x}{x} \frac{y}{1 - 0.5y}$$

i.e.

$$\left(\frac{1}{y} - \frac{1}{2}\right)\frac{dy}{dx} = -\frac{3}{4}\frac{1}{x} + \frac{1}{4}$$

By integration:
$\frac{3}{4}\ln x + \ln y - \frac{1}{2}y - \frac{1}{4} = C$, $\quad$ C=constant

The trajectories are closed curves around the critical point.



The critical point is a center and is stable.
The evolution of the predator-prey system is cyclic.

Study of general Lotka-Volterra equations

$$\frac{dx}{dt} = x(a - \alpha y) \qquad \leftarrow F(x, y)$$

$$\frac{dy}{dt} = y(-c + \gamma x) \qquad \leftarrow G(x, y)$$

The critical points are $(0, 0)$ and $(c/\gamma, a/\alpha)$.
The system is almost linear near each of its critical points.

Case of $(0, 0)$**:** Extinction of both species.

The approximating linear system is

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & -c \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Eigenvalues are $\lambda_1 = a$, $\lambda_2 = -c$.
So $(0, 0)$ is a saddle point (unstable) for both the linear and the nonlinear systems.

Case of $(c/\gamma, a/\alpha)$**:** survival of both species.

The approximating linear system is

$$\frac{d}{dt}\begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} 0 & -\alpha c/\gamma \\ \gamma a/\alpha & 0 \end{pmatrix}\begin{pmatrix} u \\ w \end{pmatrix}$$

with $u = x - c/\gamma$ and $w = y - a/\alpha$.
The eigenvalues are $\lambda_1 = i\sqrt{ac}$ and $\lambda_2 = -i\sqrt{ac}$.
So $(c/\gamma, a/\alpha)$ is a stable center for the linear system.

To determine the trajectories of the linear approximation:

$$\frac{dw}{du} = \frac{dw/dt}{du/dt} = -\frac{(\gamma a/\alpha)u}{(\alpha c/\gamma)w},$$

which can be rewritten as:

$$\gamma^2 au\frac{dw}{du} = -\alpha^2 cw$$

. Thus:

$$\gamma^2 au^2 + \alpha^2 cw^2 = k, \qquad k = \text{constant} \geq 0\,.$$

The trajectoires are concentric ellipses centered at the critical point.

For the nonlinear system, one can proceed as in the example:

We know that:

$$\frac{dx}{dt} = x(a - \alpha y) \qquad \longleftarrow F(x, y)$$

$$\frac{dy}{dt} = y(-c + \gamma x) \qquad \longleftarrow G(x, y)$$

So

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{(-c + \gamma x)}{x} \; \frac{y}{(a - \alpha y)}$$

which is a separable differential equation.

By integration one gets the implicit solution

$$a \ln y - \alpha y + c \ln x - \gamma x = C$$

where C is a constant.

The graph of a trajectory corresponding to each fixed value of C is a closed curve surrounding the critical point $(c/\gamma, a/\alpha)$. This critical point is a center.

The predator-prey system presents a cyclic variation.

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