CHAPTER 1

1.1 : MODELS & SOLUTIONS

Differential equations are tools which show up in many different fields (engineering, physics, biology, economics ...) because they provide mathematical models of physical processes and phenomena.

- What is a differential equation?

DEF 1 A differential equation (= DE) is an equation which involves an unknown function and its derivatives

EX 1 $y' + 2y = 6$; here the unknown function is y
If y is a function of the variable t, then $y' = \frac{dy}{dt}$

$\frac{dy}{dt} + 2y = 6$

or $f'(t) + 2f(t) = 6$ if $y = f(t)$

three equivalent ways of writing the same diff equation

EX 2 A DE may involve other known functions of the same variable t or functions of y and its derivatives

$\frac{dy}{dt} + 3e^t y = 2 + t$ or $y'' + y^3(y')^2 = \sin t$

DEF 2 A solution of a DE is a differentiable function satisfying the DE

EX 1 $y = e^{-2t} + 3$ is a solution. Indeed; $y' = (e^{-2t}+3)' = -2e^{-2t}$

and if we substitute y, y' in the DE, then the equality is true

$$y' + 2y = -2e^{-2t} + 2(e^{-2t} + 3) = -2e^{-2t} + 2e^{-2t} + 6 = 6$$

REMARKS :
- A solution is a function
- There might be more than one solutions. Often the solution is a class of functions

EX 1 If C is any constant, then $y = Ce^{-2t} + 3$ is a solution of $y' + 2y = 6$. We will see that there are no other solutions.

(2) DE's as mathematical models :

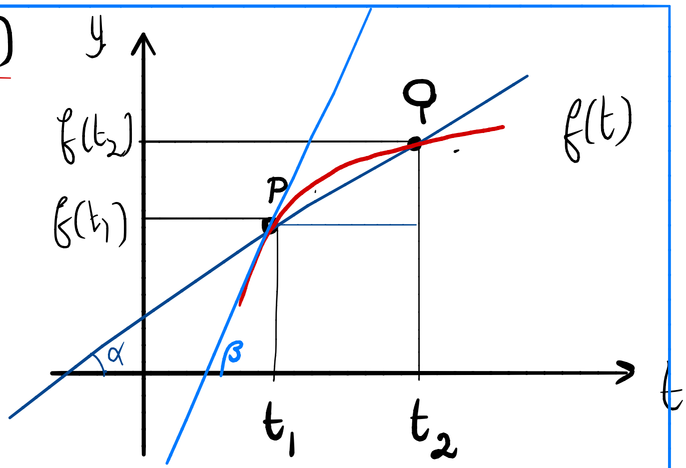
EXAMPLE (NEWTON'S LAW OF COOLING)

The temperature of an object adjusts to the temperature of the environment where it is placed (if it is hotter than the environment its temperature decreases; if it is colder then its temperature increases). The change of temperature is due to the fact that heat is flowing out or in the object; and the heat flow is caused by the temperature difference between the object and the environment. The rate at which a hot object in a cold room cools off also depends on how hot it is. For small objects put in a large environment, this behaviour is precised by the following law

NEWTON'S LAW OF COOLING (OR HEATING) : The rate at which an object changes temperature is negatively proportional to the difference between its temperature and the surrounding temperature

To express this law by a DE, we need first to recall what we mean by rate of change of a function (= how fast is a function changing)

RATE OF CHANGE OF A FUNCTION: $y=f(t)$



The average rate of change of y wrt t over the interval $[t_1,t_2]$

is $$\frac{\Delta y}{\Delta t} = \frac{f(t_2)-f(t_1)}{t_2-t_1} = \tan\alpha$$ ← ratio of change of y over change of t

It represents the slope of the secant line to the graph of f through the points P and Q

By taking the average rate of change over smaller and smaller intervals (i.e. by letting $t_2 \to t_1$) we obtain the (instantaneous) rate of change of y with respect to t at $t=t_1$

$$\lim_{\Delta t\to 0}\frac{\Delta y}{\Delta t} = \lim_{t_2\to t_1}\frac{f(t_2)-f(t_1)}{t_2-t_1} = f'(t_1) = \tan\beta$$ ← the derivative at t_1 [if the limit exists]

It represents the slope of the tangent to the graph of f at P

In Newton's law states: if

$u(t)$ = temperature of the object at time t

$T(t)$ = — " — of the environment (ambient temperature)

then $$\underbrace{\frac{du}{dt}}_{\text{the rate of change of } u(t)} = -k(u-T)$$ [← equality as functions of t]

where $k>0$ is a constant

REMARK

Newton's law is an example of DE that describes a <u>dynamical system</u>.

A dynamical system is composed of:

- a <u>system</u>; this means that we are observing a phenomenon which behaves according to a set of laws (the phenomenon may be mechanical, biological, social, ...)
- <u>dynamics</u>, i.e. the system evolves in time

Our task is to <u>predict</u> and <u>characterize</u> (as much as possible) the long-term behavior of the dynamical system and how it changes.

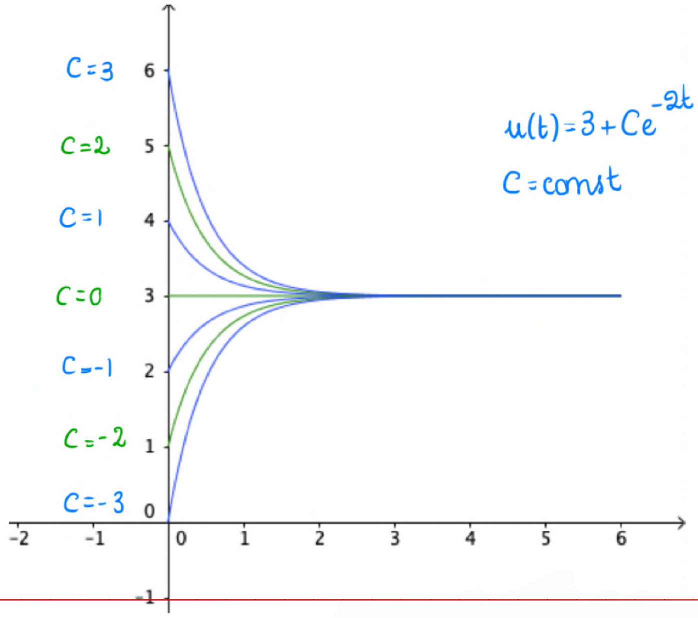
REM: If $k=2$ and $T_0=3$, we obtain $u' = -2(u-3)$
(see ex. 1)

The general solution $u = T_0 + Ce^{-kt}$ represents an infinite family of curves in the (t,u) plane, called <u>integral curves</u> of the DE $(*)$. Each integral curve corresponds to a fixed value of C

$u(t) = 3\,Ce^{-2t}$

$\lim\limits_{t\to+\infty} 3 + Ce^{-2t} = 3 \; (=T_0)$

$C = u(0) - 3 = u(0) - T_0$



For any solution u of $(*)$ [i.e. for any choice of C]

$\lim\limits_{t\to+\infty} T_0 + Ce^{-kt} = T_0$ (because $k>0$)

The temperature of the object tends to be equal to the that of the surroundings. This is the <u>long-time behaviour</u> of the system.

If $t=0$: $u(0) = T_0 + C$, i.e. $C = u(0) - T_0$ is the difference between the temperature of the object and the ambient temperature.

If $C>0$ [$C<0$] (the object is warmer [cooler] than the ambient), then the curve decreases [increases] (the temperature of the obj. decreases [increases])

The slope of the curve (which is the rate of change of $u(t)$) is proportional to $u - T_0$, in particular to $C = u(0) - T_0$ at time $t=0$.