Chapter 8: Numerical methods

Section 8.1: Euler's Method

Main Topics:

- **Direction fields,**
- **tangent lines,**
- **piecewise linear approximations of solutions**
- **Euler's Method**

Direction fields

Back to Chapter 1:

Consider the first order differential equation $\quad y' = f(t, y)$.

A solution is a function $y = \phi(t)$, defined for t in a suitable interval, which satisfies the differential equation.
This means that $\phi'(t) = f(t, \phi(t))$ for all t.

A solution describes a curve $\phi : t \mapsto y(t)$ (usually called a **solution curve**) in the ty-plane.

The set of all solution curves fills up the ty-plane.

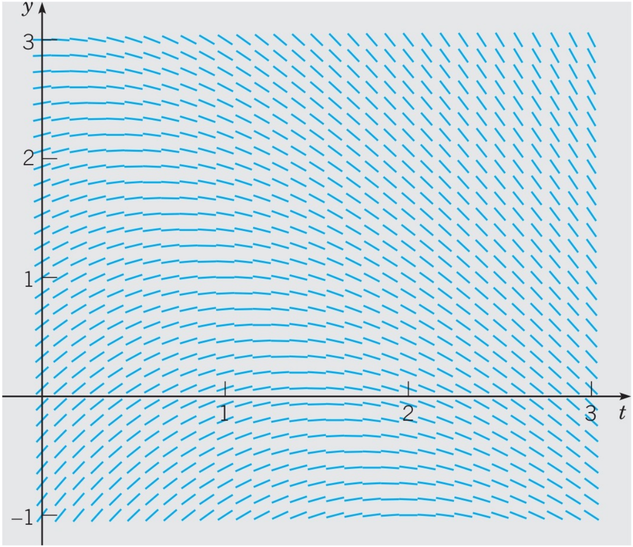
If a solution curve passes through the point (t, y), then $y = \phi(t)$.
The slope of the tangent vector to the curve $\phi : t \mapsto (t, \phi(t))$ at (t, y) is $\phi'(t)$, i.e. $f(t, \phi(t))) = f(t, y)$.

A direction field is constructed by:

- fixing a grid of points (t, y) in the ty plane, and
- drawing a short vector parallel to $(1, f(t, y))$ at each point (t, y) of the grid.

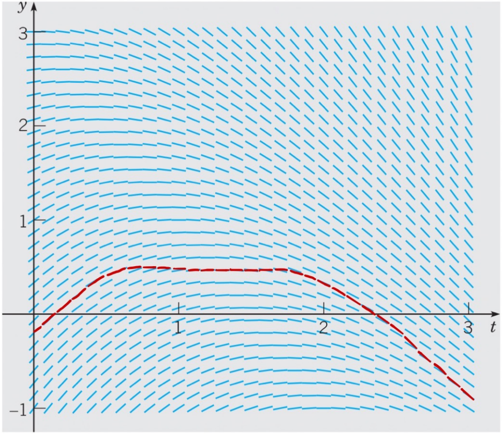
Drawing a direction field, one can visualize the behavior of the solutions of a differential equations and guess the shape of the solution curves.

A direction field for a first order differential equation $y' = f(t, y)$:



A particular solution is specified by an initial condition $y(t_0) = y_0$.
The corresponding solution $\phi : t \mapsto y(t)$ is defined on some interval containing the intial time t_0. So its solution curve passes through the point (t_0, y_0) of the ty-plane.

If (t_0, y_0) is one of the points of the grid, the corresponding vector gives the initial velocity vector. (If not, we can choose a point of the grid as close as possible to (t_0, y_0) to approximate the initial velocity vector.)



We can guess the shape of the solution curve starting at (t_0, y_0) by linking line segments for consecutive t-values of the grid.

The result is a piecewise linear approximation of the solution.

The procedure is fairly accurate provided the grid is sufficiently fine.

Euler's method

Euler's method provides a piecewise linear approximation of a solution of a first order differential equation by carrying out the linking of tangent lines in a systematic way.

Leonhard Euler (1707-1783),
a Swiss mathematician, physicist, astronomer, geographer and engineer.

The "most prolific" mathematician all the times (his collected works fill out 92 volumes).

Consider the initial value problem $y' = f(t, y)$ with initial condition $y(t_0) = y_0$.

Let $y = \phi(t)$ be the solution, defined on an interval I containing t_0.
We want to construct a linear approximation of $y = \phi(t)$ **without knowing** $\phi(t)$.

- Suppose we have chosen a sequence of points in I:

$$t_0 < t_1 < t_2 < \cdots < t_n < \cdots$$

- The equation of the tangent line to the curve $t \mapsto \phi(t)$ at (t_0, y_0) is

$$y = y_0 + f(t_0, y_0)(t - t_0)$$

  On $[t_0, t_1]$, we approximate the solution $\phi(t)$ by

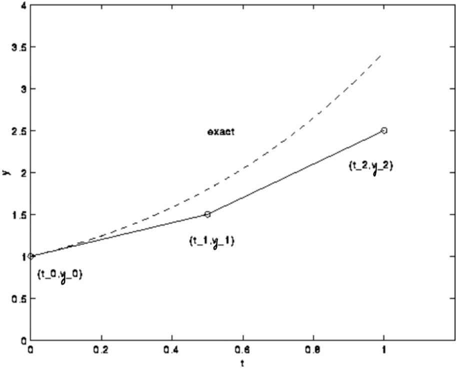
$$y(t) = y_0 + f(t_0, y_0)(t - t_0)\,.$$

  This is the equation of the line through (t_0, y_0) with the slope $f(t_0, y_0)$.

  Its value at $t = t_1$ is $y_1 = y_0 + f(t_0, y_0)(t_1 - t_0)$. It is an approximation of $\phi(t_1)$.
- Since we do not know the value of $\phi(t_1)$, at the next step we **suppose** that the value of the solution at $t = t_1$ is y_1.
- On $[t_1, t_2]$, we approximate the solution $\phi(t)$ with the line through (t_1, y_1) with the slope $f(t_1, y_1)$:

$$y = y_1 + f(t_1, y_1)(t - t_1)$$

  Its value at $t = t_2$ is $y_2 = y_1 + f(t_1, y_1)(t_2 - t_1)$. It is an approximation of $\phi(t_2)$.



- We iterate:

  On $[t_n, t_{n+1}]$, we approximate the solution $\phi(t)$ with the line through (t_n, y_n) with the slope $f(t_n, y_n)$, i.e.

  $$y = y_n + f(t_n, y_n)(t - t_n)$$

  Its value at $t = t_n$ is $y_{n+1} = y_n + f(t_n, y_n)(t_{n+1} - t_n)$. It is an approximation of $\phi(t_{n+1})$.

Euler's Formula

Suppose the solution of the initial value problem

$$\begin{cases} \dfrac{dy}{dt} = f(t, y) \\ y(t_0) = y_0 \end{cases} \tag{12}$$

is denoted $y = \phi(t)$ and you have a sequence of points $t_0 < t_1 < t_2 < \cdots < t_n < \cdots$. For $n = 0, 1, 2, \ldots,$ we have the following:

Approximation of $y = \phi(t)$ *at* $t = t_{n+1}$:

$$y_{n+1} = y_n + f(t_n, y_n)(t_{n+1} - t_n). \tag{13}$$

Linear approximation of $\phi(t)$ *on the interval* $[t_n, t_{n+1}]$:

$$y(t) = y_n + f(t_n, y_n)(t - t_n). \tag{14}$$

Special case: If a uniform step size h is used, then $t_{n+1} - t_n = h$, for all n, and so Eq. (13) simplifies to

$$y_{n+1} = y_n + hf(t_n, y_n).$$

The above approximation process for the initial value problem

$$\frac{dy}{dt} = f(t, y) \text{ with } y(t_0) = y_0$$

is known as **Euler's method of approximation** (or the tangent line method of approximation).

If one assumes for simplicity that step size h is constant
(which is not necessary to perform this method!),
Euler's method involves the repeated evaluation of the expressions

$$\begin{aligned} t_{n+1} &= t_n + h \\ y_{n+1} &= y_n + hf(t_n, y_n) \end{aligned}$$

for $n = 0, 1, 2, \cdots$.

Euler's mthod is easy to implement numerically and sufficiently accurate for simple differential equations, provided the step size h is small enough.

Example:

Apply Euler's method on the interval $[0, 1]$ with uniform step size $h = 0.25$ to the initial value problem:

$$y' = t + 2y \quad \text{with} \quad y(0) = 0$$

Remark: the exact solution of this IVP is

$$y(t) = 0.25e^{2t} - 0.5t - 0.25$$

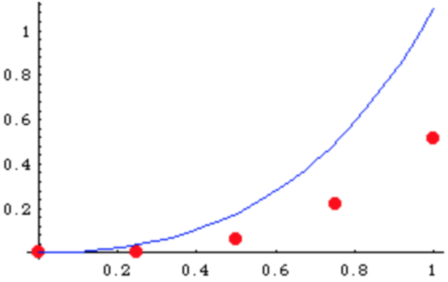
(This formula will be useful to see how accurate is Euler's approximation according to the choice of the step size.)

Here $t_0 = 0$, $y_0 = 0$, $h = 0.25$ and $f(t, y) = t + 2y$. So

$$y_{n+1} = y_n + 0.25(t_n + 2y_n)$$

$t_0 = 0$, $y_0 = 0$, $h = 0.25$ and $y_{n+1} = y_n + 0.25(t_n + 2y_n)$.

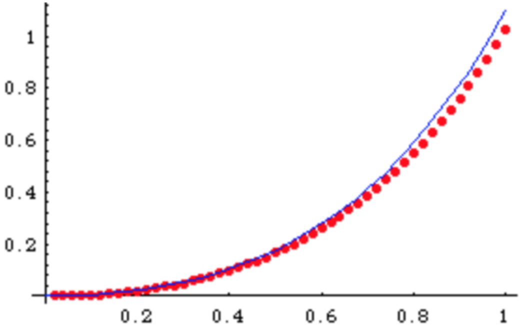
0	0.00	0.000000
1	0.25	0.000000
2	0.50	0.062500
3	0.75	0.218750
4	1.00	0.515625



The blue curve is the exact solution.

Our numerical solution (red dots) is inaccurate is because our step size is too large.

To improve the approximation, we must reduce the step size:
New step size $h = 0.02$.



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Conclusion: The accuracy of the numerical solution with $h = 0.02$ is now much higher than before.

The smaller is the step size, the higher is the accuracy of the numerical solution.

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The smaller is the step size, the higher is the accuracy of the numerical solution.

.... so is the amount of work needed!

Computers turn out to be unavoidable to get reasonable numerical solutions.