Section 8.2: Accuracy of numerical methods

Main Topics:

- **Accuracy of Euler's method,**
- **Error terms**
- **Taylor expansions**

Recall Euler's method for the initial value problem

$$y'(t) = f(t, y) \quad \text{with initial condition } y(t_0) = y_0$$

Goal: to approximate the solution $y = \phi(t)$ by a *piecewise linear function* on some interval $[a, b]$, where $a = t_0$.

Tools: the given function $f(t, y)$ and the given initial condition $y(t_0) = y_0$.

Grid: choose mesh points

$$a = t_0 < t_1 < t_2 < \cdots < t_N = b$$

and split the interval $[a, b]$ as

$$[a, b] = [a = t_0, t_1] \cup [t_1, t_2] \cup \cdots \cup [t_{N-1}, t_N = b]$$

To simplify, all intervals will be supposed to have equal length h.

Euler's algorithm: for $n = 1, 2, \cdots, N$, the approximation y_n of $\phi(t_n)$ is recursevely defined, according to

$$t_{n+1} = t_n + h$$
$$y_{n+1} = y_n + hf(t_n, y_n)$$

Example:

$$y'(t) = 1 - t + 4y$$
$$y(0) = 1$$

So: $f(t, y) = 1 - t + 4y, \quad t_0 = 0 \quad \text{and} \quad y_0 = 1.$

Take $[a, b] = [t_0, b] = [0, 2]$ and compare Euler's approximations for different choices of the step size h with the exact solution

$$y = \phi(t) = \frac{1}{4}t - \frac{3}{16} + \frac{19}{16}e^{4t}.$$

A comparison of results for the numerical solution of $\frac{dy}{dt} = 1 - t + 4y$, $y(0) = 1$ using the Euler method for different step sizes h.

t	$h = 0.05$	$h = 0.025$	$h = 0.01$	$h = 0.001$	Exact
0.0	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.1	1.5475000	1.5761188	1.5952901	1.6076289	1.6090418
0.2	2.3249000	2.4080117	2.4644587	2.5011159	2.5053299
0.3	3.4333560	3.6143837	3.7390345	3.8207130	3.8301388
0.4	5.0185326	5.3690304	5.6137120	5.7754844	5.7942260
0.5	7.2901870	7.9264062	8.3766865	8.6770692	8.7120041
1.0	45.588400	53.807866	60.037126	64.382558	64.897803
1.5	282.07187	361.75945	426.40818	473.55979	479.25919
2.0	1745.6662	2432.7878	3029.3279	3484.1608	3540.2001

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Accurate if $h = 0.001$, but... 2000 steps to move from $t = 0$ to $t = 2$.

Errors in numerical approximations

Several sources of errors (step size, rounding off, ...)

So several types of errors have to be considered.

Set:
$\phi(t_n)$ = exact value of solution at time t_n
y_n = approximation of solution at time t_n by Euler's algorithm
Y_n = rounded-off value for y_n (finite number of digits)

Local truncation error: e_n
It is the difference between the exact solution and its numerical approximation at the single step t_n *assuming* that all steps from 1 to $n-1$ are correct.
So it is equal to $\phi(t_n) - y_n$ **if** we assume that $\phi(t_1) = y_1$,..., $\phi(t_{n-1}) = y_{n-1}$.

Global truncation error: $E_n = \phi(t_n) - y_n$ (without any assumption on the previous steps)
It is the cumulative effect of the local truncation errors at each step, up to the n-th step.

Round-off error: $R_n = y_n - Y_n$
Result of rounding-off of the correct value y_n due to computer limitations.

Total error at the step t_n: $\phi(t_n) - Y_n$
Bound for the total error at t_n:

$$|\phi(t_n) - Y_n| = |(\phi(t_n) - y_n) + (y_n - Y_n)| \leq |\phi(t_n) - y_n| + |y_n - Y_n| \leq |E_n| + |R_n|$$

Bound for the total error at t_n:

$$\text{Absolute value of total error at } t_n = |\phi(t_n) - Y_n| \leq |E_n| + |R_n|$$

So:

$$\text{estimating } |E_n| \text{ and } |R_n| \Rightarrow \text{estimating the accuracy of Euler's method.}$$

- the round-off error R_n is hard to estimate, as more random in nature (it depends on the type of computer one uses).
- the global truncation error E_n coincide with the local truncation error e_n **if** all previous steps are exact.

This is what we do in the following.
More precisely: Suppose that $y_1 = \phi(t_1), \ldots, y_n = \phi(t_n)$. Then we estimate

$$|E_{n+1}| = |e_{n+1}| = |\phi(t_{n+1}) - y_{n+1}|$$

Estimation errors by Taylor expansions

Taylor expansion of ϕ about t_n:

$$\phi(t_n + h) = \phi(t_n) + \phi'(t_n)h + \frac{1}{2}\phi''(\bar{t}_n)h^2$$

where $\bar{t}_n$ is some point in the interval $(t_n, t_n + h)$.

Recall Euler's approximation

$$y_{n+1} = y_n + hf(t_n, y_n)$$

Substracting both equations, we get:

$$\phi(t_{n+1}) - y_{n+1} = (\phi(t_n) - y_n) + h(f(t_n, \phi(t_n)) - f(t_n, y_n)) + \frac{1}{2}\phi''(\bar{t}_n)h^2$$

We now apply this formula to the true solution, i.e. take $y_n = \phi(t_n)$, and get:

$$e_{n+1} = \frac{1}{2}\phi''(\bar{t}_n)h^2$$

So, the local truncation error is proportional to the square h^2 of the step size.

We found: $e_{n+1} = \frac{1}{2}\phi''(\bar{t}_n)h^2$
Problem: how to compute $\phi''(\bar{t}_n)$ if we do not know $\phi(t)$?

Recall that $\phi'(t) = f(t, \phi(t))$.

By the chain rule, we see that:

$$\begin{aligned} \phi''(t) &= f_t(t, \phi(t)) + f_y(t, \phi(t))\phi'(t) \\ &= f_t(t, \phi(t)) + f_y(t, \phi(t))f(t, \phi(t)) \end{aligned}$$

But there still the issue of $\bar{t}_n$.

One option is to consider the maximum possible value M of $| \phi''(t) |$ (i.e. of the right-hand side of the above equation, which we can compute) on the full interval $[a, b]$. Then one has

$$| e_n | \leq \frac{1}{2} M h^2$$

Conversely: if we want the absolute value of the local truncation error to be no greater than some fixed value ϵ, we have to choose a step size h such that:

$$h \leq \sqrt{2\epsilon / M}$$

Example:

Back to the example $\frac{dy}{dt} = 1 - t + 4y$ with initial condition $y(0) = 1$

The exact solution $\phi(t) = \frac{1}{16}(4t - 3 + 19e^{4t})$ gives us the best estimate for ϕ''.

We have $\phi''(t) = 19e^{4t}$ and hence

$$e_{n+1} = \frac{19e^{4\bar{t}_n}h^2}{2} \text{ for some } \bar{t}_n \in (t_n, t_n + h)$$

Suppose step size is $h = 0.05$.

The error in first step is

$$e_1 = \phi(t_1) - y_1 = \frac{19e^{4\bar{t}_0}0.0025}{2} \text{ for some } 0 < \bar{t}_0 < 0.05$$

Now since $e^{4\bar{t}_0} < e^{0.2}$ we get that

$$e_1 \leq \frac{19e^{0.2}0.0025}{2} \simeq 0.02901$$

Note also that $e^{4\bar{t}_0} > 1$, and thus

$$e_1 > 19 \times 0.0025/2 = 0.02375$$

In other words, we have bounded the local error

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Note that the actual error is 0.02542.

So our bounds are not too bad!

We can proceed similarly for the next steps $t_2, t_3, \cdots$.
We find that the error becomes progressively worse with increasing t.