Section 8.3: Improved Euler and Runge-Kutta methods

Main Topics:

Two variations of Euler's method to improve its accurancy:

- **the improved Euler's method**,
- **Runge-Kutta's method**

As for Euler's method, we consider the initial value problem

$$y'(t) = f(t, y) \quad \text{with initial condition } y(t_0) = y_0$$

Goal: to approximate the solution $y = \phi(t)$ by a *piecewise linear function* on some interval $[t_0, b]$ by a recursive method using:
the given function $f(t, y)$ and the given initial condition $y(t_0) = y_0$.

Grid: choose mesh points $t_0 < t_1 < t_2 < \cdots < t_N = b$
To simplify, all intervals will be supposed to have equal length h.

Key remark:
If $y = \phi(t)$ is a solution, then $\phi'(t) = f(t, \phi(t))$, i.e.

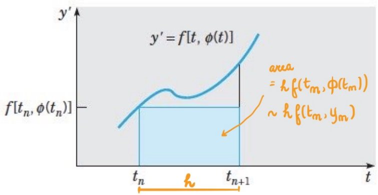
$$\phi(t_{n+1}) - \phi(t_n) = \int_{t_n}^{t_{n+1}} f(t, \phi(t))\, dt$$

The RHS is the area of the region between the graph of $t \mapsto f(t, \phi(t))$ and the t-axis above the interval $[t_n, t_{n+1}]$.

The improved Euler's method

Euler's algorithm:

$$t_{n+1} = t_n + h$$
$$y_{n+1} = y_n + hf(t_n, y_n)$$



The improved Euler's method

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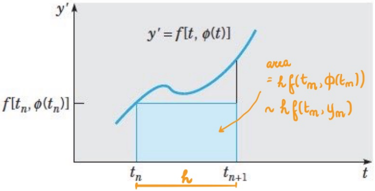
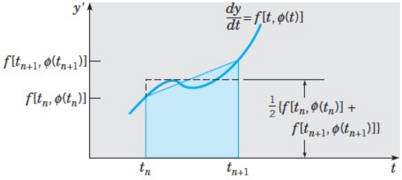


Improved Euler's algorithm:

Idea:

$$t_{n+1} = t_n + h$$
$$y_{n+1} = y_n + h\frac{f(t_n, y_n) + f(t_{n+1}, y_{n+1})}{2}$$



Idea:

$$t_{n+1} = t_n + h$$
$$y_{n+1} = y_n + h\frac{f(t_n, y_n) + f(t_{n+1}, y_{n+1})}{2}$$

Problem:
the unknown y_{n+1} appears as an argument of f on the RHS. It is an implicit equation for y_{n+1}.

Solution: Remplace t_{n+1} and y_{n+1} inside $f(t_{n+1}, y_{n+1})$ by their values as computed in Euler's algorithm:

$$t_{n+1} = t_n + h \qquad \text{and} \qquad y_{n+1} = y_n + hf(t_n, y_n)\,.$$

Get the **improved Euler's method** (or **Heun's method**):

$$t_{n+1} = t_n + h$$
$$y_{n+1} = y_n + h\frac{f(t_n, y_n) + f(t_n + h, y_n + hf(t_n, y_n))}{2}$$

The Runge-Kutta method

Idea: Modify Euler's iteration: $y_{n+1} = y_n + hf(t_n, y_n)$
by replacing $f(t_n, y_n)$ by a weighted average of values of $f(t, y)$ on the interval $[t_n, t_{n+1}]$.

Runge–Kutta Method

Suppose the solution of the initial value problem

$$\begin{cases} \dfrac{dy}{dt} = f(t, y) \\ y(t_0) = y_0 \end{cases} \tag{18}$$

is denoted $y = \phi(t)$ and you have a sequence of points $t_0 < t_1 < t_2 < \cdots < t_n < \cdots$. For $n = 0, 1, 2, \ldots$, we have the following:

Approximation of $y = \phi(t)$ *at* $t = t_{n+1}$:

$$y_{n+1} = y_n + h\left(\frac{k_{n1} + 2k_{n2} + 2k_{n3} + k_{n4}}{6}\right), \tag{19}$$

where

$$\begin{cases} k_{n1} = f(t_n, y_n) \\ k_{n2} = f\left(t_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_{n1}\right) \\ k_{n3} = f\left(t_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_{n2}\right) \\ k_{n4} = f(t_n + h, y_n + hk_{n3}). \end{cases} \tag{20}$$

Linear approximation of $\phi(t)$ *on the interval* $[t_n, t_{n+1}]$:

$$y(t) = y_n + f(t_n, y_n)(t - t_n). \tag{21}$$

Special case: If $f(t, y) = f(t)$, then Eq. (19) simplifies to

$$y_{n+1} = y_n + \frac{h}{6}\left[f(t_n) + 4f\left(t_n + \frac{h}{2}\right) + f(t_n + h)\right]. \tag{22}$$