Section 1.1: Mathematical models and solutions

Main Topics:

- **Mathematical models**
- **Newton's Law of cooling**
- **Analytic methods**
- **Initial Value Problems (IVP)**

Example: Newton's Law of Cooling

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$$\frac{du}{dt} = -k(u - T) \qquad \text{or} \qquad u' = -k(u - T) \tag{1}$$

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Equation (1) is an example of a *differential equation*.

Definition

A **differential equation** (**DE** for short) is an equation involving a function and its derivatives.

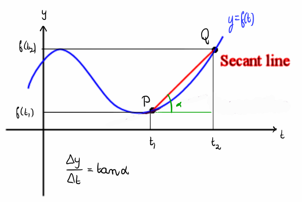
The rate of change of a differentiable function $y = f(t)$

The **average rate of change** of y with respect to t over the interval $[t_1, t_2]$ is

$$\frac{\Delta y}{\Delta t} = \frac{f(t_2) - f(t_1)}{t_2 - t_1}$$

It is the slope of the secant line to the graph of f thorugh P and Q.

average rate of change = slope of the secant line



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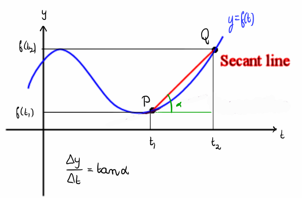

By taking the average rate of change over smaller and smaller intervals (i.e. by letting $t_2 \to t_1$) the secant line becomes the tangent line.
We obtain the **(instantaneous) rate of change of y with respect to t at t_1**:

$$\frac{dy}{dt} = \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} = \lim_{t_2 \to t_1} \frac{f(t_2) - f(t_1)}{t_2 - t_1} = f'(t_1)$$

It is the slope of the secant line to the graph of f at P.

rate of change at t_1 = slope of the tangent at P=$f'(t_1)$

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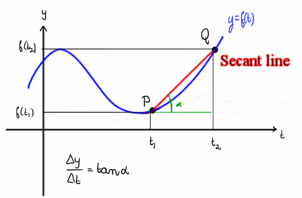

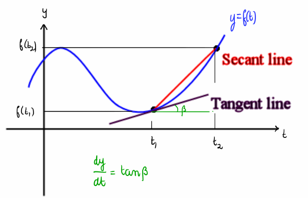

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Remarks:

- A solution of a DE is a function
- There might be more than one solution. Often the solution is a class of functions.

The solution of the example can be found analytically (method of separation of variables. See Chapter 2).

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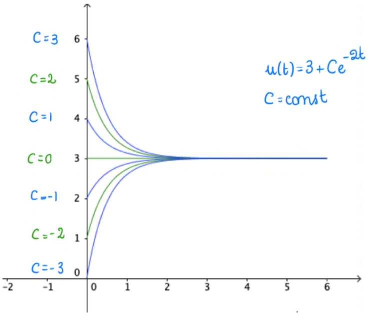
General solution: $u(t) = 3 + Ce^{-2t}$, where $C \in \mathbb{R}$.

For each choice of C we get a solution: $u(0) = 3 + C$

Its graph is a curve in the (t, u)-plane, called an **integral curve** of the DE.



Long time behavior: $\lim_{t\to+\infty} u(t) = 3$.

Initial Value Problems (IVP)

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that satisfies the initial condition $u(0) = 1$.

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The **initial condition** $u(0) = 1$ singles out a unique solution among the infinetely many solutions $u(t) = 3 + Ce^{-2t}$, where $C \in \mathbb{R}$.

Indeed

$$1 = u(0) = 3 + Ce^{-2\cdot 0} \qquad \text{yields} \qquad C = -2\,.$$

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The initial condition could be imposed at some other time t.
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Dynamical Systems

Newton's Law is one example of an equation that describes a *dynamical system*.

A dynamical system is composed of:

- A *system*: Which means that we are observing a phenomenon which behaves according to a set of laws.
- The phenomenon may be mechanical, biological, social, etc.
- *Dynamics*: the system evolves in time.

It is our task to *predict* and *characterize* (as much as possible) the long-term behavior of the dynamical system and how it changes.

This leads us to the use of derivatives and the geometric methods we explore for the rest of the course.