Section 1.2: Qualitative methods: phase lines and direction fields

Main Topics:

- **Mathematical models and direction fields**,
- **Diagrams**

Objectives:

- Determine and classify **equilibrium solutions**
- Sketch **direction fields** and **phase lines**
- Sketch solution curves of **autonomous DE's** based on a qualitative analysis

Direction Fields

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- In the next slide, we explore one qualitative method: the **direction field**, and how we can construct it.

Direction Field Example

Example: recall Newton's Law of Cooling, $\frac{du}{dt} = -k(u - T)$.

If $T = 1$ and $k = 2$. Then

$$\frac{du}{dt} = -k(u - T) = 2 - 2u$$

If $u = 0$, then $\frac{du}{dt} =$ ___

If $u = 1$, then $\frac{du}{dt} =$ ___

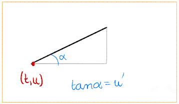
If $u = 2$, then $\frac{du}{dt} =$ ___

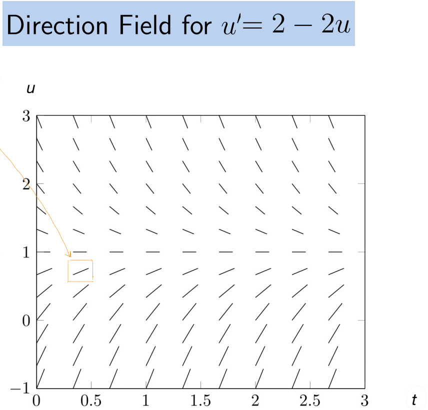
If $u = 3$, then $\frac{du}{dt} =$ ___

We use this information to sketch a set of line segments that characterize $u'(t)$ for a particular k and T.

Direction Field for $u'(t) = 2 - 2u$

We can plot line segments for a set of points (t, u), whose slope are determined by $u'(t)$.





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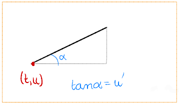

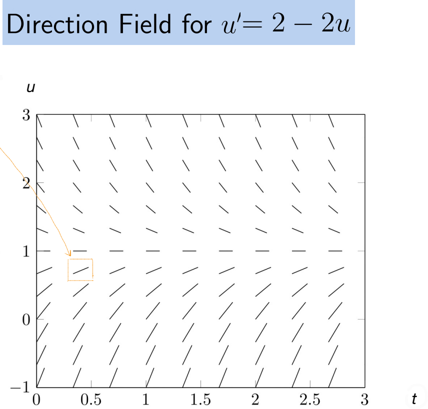

- For what values of u is $\dfrac{du}{dt} = 0$?

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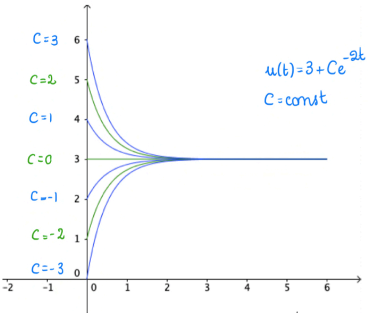
Example: $\dfrac{du}{dt} = -2(u-3)$ (Newton's law of cooling with $k=2$ and $T=3$)

Some graphs of solutions in the (t,u)-plane (**integral curves**).



Long time behavior: $\lim_{t\to+\infty} u(t) = 3$.

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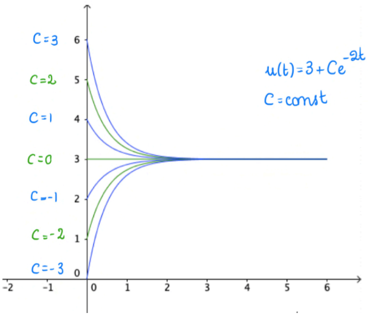

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$\frac{du}{dt} = -2(u-3)$ is an autonomous DE, of the form $\frac{du}{dt} = f(u)$, with $f(u) = -2(u-3)$

The equilibrium solutions of $\frac{du}{dt} = -2(u-3)$ are the solutions to $f(u) = 0$.

Hence, here there is a unique equilibrium solution, namely $u = 3$.

Phase Portraits and Stability

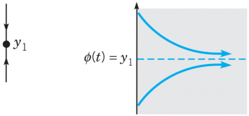
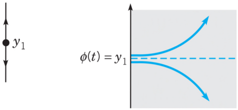
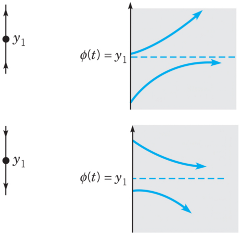
Example:

Let $f(y) = y(1 - y^2)$.

Consider the differential equation $\dfrac{dy}{dt} = f(y)$ for $t \geq 0$.

- Determine the equilibrium points.
- Sketch $f(y)$ vs y.
- Use the information from the questions above to sketch the **phase line** for the DE.
- Sketch a few solution curves, or **integral curves** for the DE.

Classification of equilibrium points

Phase Line / Sample Solution Curves	Verbal Interpretation	Classification y_1 is a(n) ...
	Solution curves passing through points whose y-values close to y_1 on either side tend toward y_1 asymptotically as $t \to \infty$.	**asymptotically stable equilibrium point**
	Solution curves passing through points whose y-values close to y_1 on either side tend away from y_1 as $t \to \infty$.	**unstable equilibrium point**
	Solution curves tend away from y_1 if they pass through points whose y-values are close to y_1 on one side, but they tend toward y_1 asymptotically as $t \to \infty$ if they pass through points whose y-values are close to y_1 on the opposite side.	**semistable equilibrium point**

J.Brannan & W. Joyce, Differential Equations, Table 1.2.2

Example:

For each of the following differential equations $\frac{dy}{dt} = f(y)$:

- Scketch the graph of $f(y)$ versus y.
- Determine the critical (equilibrium) points.
- Draw the phase line and sketch some graphs of solutions.
- Classify each of the critical points.

(1) $f(y) = 2 - 2y$

(2) $f(y) = (y - 1)^2$