Section 1.3: Classification of Differential Equations

Classification:

- provides an organized framework for the subject,
- helps us to choose a method for solving a given DE.

The basic criteria for classification:

- The number of independent variables
- The order of the DE
- The number of unknown functions

- Ordinary Differential Equation (ODE): the unknown function depends on a single independent variable.
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- $\frac{dy}{dt} = y + t$ or $\frac{d^2y}{dt^2} = 3\cos y$ or $\frac{d^2y}{dt^2} + 2y\frac{dy}{dt} = y$ are ODE's The unknown function is y(t). The unique independent variable is t.

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Examples of PDE's

- The heat equation $\frac{\partial}{\partial t}u(x,t) = C\frac{\partial^2 u(x,t)}{\partial x^2}$ The unknown function is u(x,t) (the temperature of a metal rod). Two independent variables: t=time, x=position along the rod
- $\frac{\partial}{\partial t}u + u\frac{\partial u}{\partial x} = 0$ is a PDE. The unknown function is u(x,t); two independent variables: t and x.

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This course focuses on ODE's.



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Remark: This course mostly focuses on ODE's of order 1 and 2.

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A system of three ODE's:

$$\begin{cases} \frac{dx_1}{dt} = 2x_1 + 3x_2 + x_3 \\ \frac{dx_2}{dt} = x_1 - x_2 \\ \frac{dx_3}{dt} = -x_1 + x_3 \end{cases}$$
 three unknown functions $x_1(t)$, $x_2(t)$ and $x_3(t)$.

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$$a_0(t)y^{(n)}(t) + a_1(t)y^{(n-1)}(t) + \cdots + a_n(t)y(t) = g(t)$$

where

- a₀(t), a₁(t), ..., a_n(t) and g(t) are functions of t which are given (called the coefficients)
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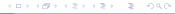
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First order linear ODEs

A first order linear ODE is of the form

$$a_0(t)\frac{dy}{dt}+a_1(t)y=g(t)$$

If $a_0(t) = 0$ for all t, there is no DE (no derivative)!

If not, for all t so that $a_0(t) \neq 0$, we can divide both sides of the DE by $a_0(t)$:

$$\frac{dy}{dt} + \frac{a_1(t)}{a_0(t)}y = \frac{g(t)}{a_0(t)}$$

Set

$$p(t) = \frac{a_1(t)}{a_0(t)}$$
 and $h(t) = \frac{g(t)}{a_0(t)}$

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For all $t \neq 0$:

$$y' + \frac{t-1}{t^3} = \frac{1}{t^2}$$
 (standard form)

