Section 1.3: Classification of Differential Equations

Classification:

- provides an organized framework for the subject,
- helps us to choose a method for solving a given DE.

The basic criteria for classification:

- The number of independent variables
- The order of the DE
- The number of unknown functions

Number of independent variables

- **Ordinary Differential Equation (ODE)**: the unknown function depends on a single independent variable.
- **Partial Differential Equation (PDE)**: the unknown function depends on more than one independent variable.

Number of independent variables

- **Ordinary Differential Equation (ODE)**: the unknown function depends on a single independent variable.
- **Partial Differential Equation (PDE)**: the unknown function depends on more than one independent variable.

Examples of ODE's

- Newton's Law of Cooling: $\frac{du}{dt} = -k(u - T)$ is an ODE.

  The unknown function is $u(t)$. The unique independent variable is t.

Number of independent variables

- **Ordinary Differential Equation (ODE)**: the unknown function depends on a single independent variable.
- **Partial Differential Equation (PDE)**: the unknown function depends on more than one independent variable.

Examples of ODE's

- Newton's Law of Cooling: $\frac{du}{dt} = -k(u - T)$ is an ODE.

  The unknown function is $u(t)$. The unique independent variable is t.
- $\frac{dy}{dt} = y + t$ **or** $\frac{d^2y}{dt^2} = 3\cos y$ **or** $\frac{d^2y}{dt^2} + 2y\frac{dy}{dt} = y$ are ODE's

  The unknown function is $y(t)$. The unique independent variable is t.

Number of independent variables

- **Ordinary Differential Equation (ODE)**: the unknown function depends on a single independent variable.
- **Partial Differential Equation (PDE)**: the unknown function depends on more than one independent variable.

Examples of ODE's

- Newton's Law of Cooling: $\frac{du}{dt} = -k(u - T)$ is an ODE.
  The unknown function is $u(t)$. The unique independent variable is t.
- $\frac{dy}{dt} = y + t$ **or** $\frac{d^2y}{dt^2} = 3\cos y$ **or** $\frac{d^2y}{dt^2} + 2y\frac{dy}{dt} = y$ are ODE's
  The unknown function is $y(t)$. The unique independent variable is t.

Examples of PDE's

- The heat equation $\frac{\partial}{\partial t}u(x,t) = C\frac{\partial^2 u(x,t)}{\partial x^2}$
  The unknown function is $u(x,t)$ (the temperature of a metal rod).
  Two independent variables: t=time, x=position along the rod
- $\frac{\partial}{\partial t}u + u\frac{\partial u}{\partial x} = 0$ is a PDE.
  The unknown function is $u(x,t)$; two independent variables: t and x.

Number of independent variables

- **Ordinary Differential Equation (ODE)**: the unknown function depends on a single independent variable.
- **Partial Differential Equation (PDE)**: the unknown function depends on more than one independent variable.

Examples of ODE's

- Newton's Law of Cooling: $\dfrac{du}{dt} = -k(u - T)$ is an ODE.
  The unknown function is $u(t)$. The unique independent variable is t.
- $\dfrac{dy}{dt} = y + t$ **or** $\dfrac{d^2y}{dt^2} = 3\cos y$ **or** $\dfrac{d^2y}{dt^2} + 2y\dfrac{dy}{dt} = y$ are ODE's
  The unknown function is $y(t)$. The unique independent variable is t.

Examples of PDE's

- The heat equation $\frac{\partial}{\partial t}u(x,t) = C\frac{\partial^2 u(x,t)}{\partial x^2}$
  The unknown function is $u(x,t)$ (the temperature of a metal rod).
  Two independent variables: t=time, x=position along the rod
- $\frac{\partial}{\partial t}u + u\frac{\partial u}{\partial x} = 0$ is a PDE.
  The unknown function is $u(x,t)$; two independent variables: t and x.

This course focuses on ODE's.

The order of an ODE

Definition

The **order** of an ODE is the highest degree derivative which appears in the equation.

The order of an ODE

Definition

The **order** of an ODE is the highest degree derivative which appears in the equation.

Examples

(1) Newton's Law of Cooling $u' = -k(u - T)$ has order

The order of an ODE

Definition

The **order** of an ODE is the highest degree derivative which appears in the equation.

Examples

(1) Newton's Law of Cooling $u' = -k(u - T)$ has order 1.

The order of an ODE

Definition

The **order** of an ODE is the highest degree derivative which appears in the equation.

Examples

(1) Newton's Law of Cooling $u' = -k(u - T)$ has order 1.

(2) The ODE $y''' + 2e^t y'' + yy' = t^4$ has order

The order of an ODE

Definition

The **order** of an ODE is the highest degree derivative which appears in the equation.

Examples

(1) Newton's Law of Cooling $u' = -k(u - T)$ has order 1.

(2) The ODE $y''' + 2e^t y'' + yy' = t^4$ has order 3.

The order of an ODE

Definition

The **order** of an ODE is the highest degree derivative which appears in the equation.

Examples

(1) Newton's Law of Cooling $u' = -k(u - T)$ has order 1.

(2) The ODE $y^{'''} + 2e^t y'' + yy' = t^4$ has order 3.

(3) $y'' + (y')^{10} = 4$ has order

The order of an ODE

Definition

The **order** of an ODE is the highest degree derivative which appears in the equation.

Examples

(1) Newton's Law of Cooling $u' = -k(u - T)$ has order 1.

(2) The ODE $y''' + 2e^t y'' + yy' = t^4$ has order 3.

(3) $y'' + (y')^{10} = 4$ has order 2.

Remark: This course mostly focuses on ODE's of order 1 and 2.

The number of unknown functions

One ODE:

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} = 3 \qquad \text{one unknown function } y(t)$$

The number of unknown functions

One ODE:

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} = 3 \qquad \text{one unknown function } y(t)$$

A system of two ODE's:

$$\begin{cases} \dfrac{dx}{dt} = 2x + 3y \\ \dfrac{dy}{dt} = x - y \end{cases} \qquad \text{two unknown functions } x(t) \text{ and } y(t)$$

The number of unknown functions

One ODE:

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} = 3 \qquad \text{one unknown function } y(t)$$

A system of two ODE's:

$$\begin{cases} \dfrac{dx}{dt} = 2x + 3y \\ \dfrac{dy}{dt} = x - y \end{cases} \qquad \text{two unknown functions } x(t) \text{ and } y(t)$$

A system of three ODE's:

$$\begin{cases} \dfrac{dx_1}{dt} = 2x_1 + 3x_2 + x_3 \\ \dfrac{dx_2}{dt} = x_1 - x_2 \\ \dfrac{dx_3}{dt} = -x_1 + x_3 \end{cases} \qquad \text{three unknown functions } x_1(t), x_2(t) \text{ and } x_3(t).$$

Linearity / Homogeneity

Definition

An **n-th order linear ODE** is an ODE of the form:

$$a_0(t)y^{(n)}(t) + a_1(t)y^{(n-1)}(t) + \cdots + a_n(t)y(t) = g(t)$$

where

- $a_0(t)$, $a_1(t)$, ..., $a_n(t)$ and $g(t)$ are functions of t which are given (called the **coefficients**)
- $y(t)$ is the unknown function.
- $y^{(k)}(t)$ denotes the k-th derivative of y.

Linearity / Homogeneity

Definition

An **n-th order linear ODE** is an ODE of the form:

$$a_0(t)y^{(n)}(t) + a_1(t)y^{(n-1)}(t) + \cdots + a_n(t)y(t) = g(t)$$

where

- $a_0(t)$, $a_1(t)$, ..., $a_n(t)$ and $g(t)$ are functions of t which are given (called the **coefficients**)
- $y(t)$ is the unknown function.
- $y^{(k)}(t)$ denotes the k-th derivative of y. **E.g.** $y^{(1)} = y'$, $y^{(2)} = y''$...

Linearity / Homogeneity

Definition

An *n***-th order linear ODE** is an ODE of the form:

$$a_0(t)y^{(n)}(t) + a_1(t)y^{(n-1)}(t) + \cdots + a_n(t)y(t) = g(t)$$

where

- $a_0(t)$, $a_1(t)$, ..., $a_n(t)$ and $g(t)$ are functions of t which are given (called the **coefficients**)
- $y(t)$ is the unknown function.
- $y^{(k)}(t)$ denotes the k-th derivative of y. **E.g.** $y^{(1)} = y'$, $y^{(2)} = y''$...

If $g(t) = 0$ for all t, we say that the linear ODE is **homogeneous**.

Linearity / Homogeneity

Definition

An n**-th order linear ODE** is an ODE of the form:

$$a_0(t)y^{(n)}(t) + a_1(t)y^{(n-1)}(t) + \cdots + a_n(t)y(t) = g(t)$$

where

- $a_0(t)$, $a_1(t)$, ..., $a_n(t)$ and $g(t)$ are functions of t which are given (called the **coefficients**)
- $y(t)$ is the unknown function.
- $y^{(k)}(t)$ denotes the k-th derivative of y. **E.g.** $y^{(1)} = y'$, $y^{(2)} = y''$...

If $g(t) = 0$ for all t, we say that the linear ODE is **homogeneous**.

An ODE which is not of the above form is called **non-linear**.

Linearity / Homogeneity

Definition

An *n***-th order linear ODE** is an ODE of the form:

$$a_0(t)y^{(n)}(t) + a_1(t)y^{(n-1)}(t) + \cdots + a_n(t)y(t) = g(t)$$

where

- $a_0(t)$, $a_1(t)$, ..., $a_n(t)$ and $g(t)$ are functions of t which are given (called the **coefficients**)
- $y(t)$ is the unknown function.
- $y^{(k)}(t)$ denotes the k-th derivative of y. **E.g.** $y^{(1)} = y'$, $y^{(2)} = y''$...

If $g(t) = 0$ for all t, we say that the linear ODE is **homogeneous**.

An ODE which is not of the above form is called **non-linear**.

Examples

(1) Newton's Law of Cooling $\frac{d}{dt}u(t) = -k(u - T)$ is linear, has order 1, is nonhomogeneous if $T \neq 0$.

Linearity / Homogeneity

Definition

An n**-th order linear ODE** is an ODE of the form:

$$a_0(t)y^{(n)}(t) + a_1(t)y^{(n-1)}(t) + \cdots + a_n(t)y(t) = g(t)$$

where

- $a_0(t)$, $a_1(t)$, ..., $a_n(t)$ and $g(t)$ are functions of t which are given (called the **coefficients**)
- $y(t)$ is the unknown function.
- $y^{(k)}(t)$ denotes the k-th derivative of y. **E.g.** $y^{(1)} = y'$, $y^{(2)} = y''$...

If $g(t) = 0$ for all t, we say that the linear ODE is **homogeneous**.

An ODE which is not of the above form is called **non-linear**.

Examples

(1) Newton's Law of Cooling $\frac{d}{dt}u(t) = -k(u - T)$ is linear, has order 1, is nonhomogeneous if $T \neq 0$.

(2) $t^3y'' + cos(t)y' = y$

Linearity / Homogeneity

Definition

An ***n*-th order linear ODE** is an ODE of the form:

$$a_0(t)y^{(n)}(t) + a_1(t)y^{(n-1)}(t) + \cdots + a_n(t)y(t) = g(t)$$

where

- $a_0(t)$, $a_1(t)$, ..., $a_n(t)$ and $g(t)$ are functions of t which are given (called the **coefficients**)
- $y(t)$ is the unknown function.
- $y^{(k)}(t)$ denotes the k-th derivative of y. **E.g.** $y^{(1)} = y'$, $y^{(2)} = y''$...

If $g(t) = 0$ for all t, we say that the linear ODE is **homogeneous**.

An ODE which is not of the above form is called **non-linear**.

Examples

(1) Newton's Law of Cooling $\frac{d}{dt}u(t) = -k(u - T)$ is linear, has order 1, is nonhomogeneous if $T \neq 0$.

(2) $t^3y'' + cos(t)y' = y$ is linear, of order 2, homogeneous.

Linearity / Homogeneity

Definition

An *n*-**th order linear ODE** is an ODE of the form:

$$a_0(t)y^{(n)}(t) + a_1(t)y^{(n-1)}(t) + \cdots + a_n(t)y(t) = g(t)$$

where

- $a_0(t)$, $a_1(t)$, ..., $a_n(t)$ and $g(t)$ are functions of t which are given (called the **coefficients**)
- $y(t)$ is the unknown function.
- $y^{(k)}(t)$ denotes the k-th derivative of y. **E.g.** $y^{(1)} = y'$, $y^{(2)} = y''$...

If $g(t) = 0$ for all t, we say that the linear ODE is **homogeneous**.

An ODE which is not of the above form is called **non-linear**.

Examples

(1) Newton's Law of Cooling $\frac{d}{dt}u(t) = -k(u - T)$ is linear, has order 1, is nonhomogeneous if $T \neq 0$.

(2) $t^3y'' + cos(t)y' = y$ is linear, of order 2, homogeneous.

(3) $yy'' + y' = 0$

Linearity / Homogeneity

Definition

An **n-th order linear ODE** is an ODE of the form:

$$a_0(t)y^{(n)}(t) + a_1(t)y^{(n-1)}(t) + \cdots + a_n(t)y(t) = g(t)$$

where

- $a_0(t)$, $a_1(t)$, ..., $a_n(t)$ and $g(t)$ are functions of t which are given (called the **coefficients**)
- $y(t)$ is the unknown function.
- $y^{(k)}(t)$ denotes the k-th derivative of y. **E.g.** $y^{(1)} = y'$, $y^{(2)} = y''$...

If $g(t) = 0$ for all t, we say that the linear ODE is **homogeneous**.

An ODE which is not of the above form is called **non-linear**.

Examples

(1) Newton's Law of Cooling $\frac{d}{dt}u(t) = -k(u - T)$ is linear, has order 1, is nonhomogeneous if $T \neq 0$.

(2) $t^3y'' + cos(t)y' = y$ is linear, of order 2, homogeneous.

(3) $yy'' + y' = 0$ is non-linear, of order 2

Linearity / Homogeneity

Definition

An **n-th order linear ODE** is an ODE of the form:

$$a_0(t)y^{(n)}(t) + a_1(t)y^{(n-1)}(t) + \cdots + a_n(t)y(t) = g(t)$$

where

- $a_0(t)$, $a_1(t)$, ..., $a_n(t)$ and $g(t)$ are functions of t which are given (called the **coefficients**)
- $y(t)$ is the unknown function.
- $y^{(k)}(t)$ denotes the k-th derivative of y. **E.g.** $y^{(1)} = y'$, $y^{(2)} = y''$...

If $g(t) = 0$ for all t, we say that the linear ODE is **homogeneous**.

An ODE which is not of the above form is called **non-linear**.

Examples

(1) Newton's Law of Cooling $\frac{d}{dt}u(t) = -k(u - T)$ is linear, has order 1, is nonhomogeneous if $T \neq 0$.

(2) $t^3y'' + cos(t)y' = y$ is linear, of order 2, homogeneous.

(3) $yy'' + y' = 0$ is non-linear, of order 2

(4) $y + y' = \sin(y + t)$

Linearity / Homogeneity

Definition

An **n-th order linear ODE** is an ODE of the form:

$$a_0(t)y^{(n)}(t) + a_1(t)y^{(n-1)}(t) + \cdots + a_n(t)y(t) = g(t)$$

where

- $a_0(t)$, $a_1(t)$, ..., $a_n(t)$ and $g(t)$ are functions of t which are given (called the **coefficients**)
- $y(t)$ is the unknown function.
- $y^{(k)}(t)$ denotes the k-th derivative of y. **E.g.** $y^{(1)} = y'$, $y^{(2)} = y''$...

If $g(t) = 0$ for all t, we say that the linear ODE is **homogeneous**.

An ODE which is not of the above form is called **non-linear**.

Examples

(1) Newton's Law of Cooling $\frac{d}{dt}u(t) = -k(u - T)$ is linear, has order 1, is nonhomogeneous if $T \neq 0$.

(2) $t^3 y'' + cos(t)y' = y$ is linear, of order 2, homogeneous.

(3) $yy'' + y' = 0$ is non-linear, of order 2

(4) $y + y' = \sin(y + t)$ is non-linear, of order 1.

First order linear ODEs

A first order linear ODE is of the form

$$a_0(t)\frac{dy}{dt} + a_1(t)y = g(t)$$

If $a_0(t) = 0$ for all t, there is no DE (no derivative)!
If not, for all t so that $a_0(t) \neq 0$, we can divide both sides of the DE by $a_0(t)$:

$$\frac{dy}{dt} + \frac{a_1(t)}{a_0(t)}y = \frac{g(t)}{a_0(t)}$$

Set

$$p(t) = \frac{a_1(t)}{a_0(t)} \quad \text{and} \quad h(t) = \frac{g(t)}{a_0(t)}$$

Then the above equation can be put in the **standard form** (or normal form)

$$\frac{dy}{dt} + p(t)y = h(t)$$

Example
$t^3y' + ty = y + t$

First order linear ODEs

A first order linear ODE is of the form

$$a_0(t)\frac{dy}{dt} + a_1(t)y = g(t)$$

If $a_0(t) = 0$ for all t, there is no DE (no derivative)!
If not, for all t so that $a_0(t) \neq 0$, we can divide both sides of the DE by $a_0(t)$:

$$\frac{dy}{dt} + \frac{a_1(t)}{a_0(t)}y = \frac{g(t)}{a_0(t)}$$

Set

$$p(t) = \frac{a_1(t)}{a_0(t)} \quad \text{and} \quad h(t) = \frac{g(t)}{a_0(t)}$$

Then the above equation can be put in the **standard form** (or normal form)

$$\frac{dy}{dt} + p(t)y = h(t)$$

Example
$t^3y' + ty = y + t$ is linear, of order 1, nonhomogeneous.

First order linear ODEs

A first order linear ODE is of the form

$$a_0(t)\frac{dy}{dt} + a_1(t)y = g(t)$$

If $a_0(t) = 0$ for all t, there is no DE (no derivative)!
If not, for all t so that $a_0(t) \neq 0$, we can divide both sides of the DE by $a_0(t)$:

$$\frac{dy}{dt} + \frac{a_1(t)}{a_0(t)}y = \frac{g(t)}{a_0(t)}$$

Set

$$p(t) = \frac{a_1(t)}{a_0(t)} \quad \text{and} \quad h(t) = \frac{g(t)}{a_0(t)}$$

Then the above equation can be put in the **standard form** (or normal form)

$$\frac{dy}{dt} + p(t)y = h(t)$$

Example
$t^3y' + ty = y + t$ is linear, of order 1, nonhomogeneous.
For all $t \neq 0$:

$$y' + \frac{t-1}{t^3} = \frac{1}{t^2} \qquad \text{(standard form)}$$