Chapter 2: First order differential equations

$$\frac{dy}{dx} = f(x, y)$$

where

- x is the independent variable
- y is the unknown function
- f is a function of two variables

Three main classes of first order DE for which there are general methods to find solutions:

- separable DE
- linear DE
- exact DE

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Definition

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$$f(x, y) = p(x)q(y)$$

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- $y' = e^x y^2$ of order 1, separable (and non-linear) $\longleftarrow$ *Let us solve this DE*
- $\frac{dy}{dx} = y + x$ of order 1, not separable (but linear)
- $\frac{dy}{dx} = e^{x-2y}$ of order 1, separable (and non-linear).

Method to solve the separable DE: $\dfrac{dy}{dx} = p(x)q(y)$

- If $a \in \mathbb{R}$ satisfies $q(a) = 0$, then $y = a$ is a constant solution of the DE. All the constant solutions of the DE arise in this way.
- To find nonconstant solutions, divide both sides of the DE by $q(y)$ and get

$$\frac{1}{q(y)} \frac{dy}{dx} = p(x)$$

- Integrate both sides with respect to x:

$$\int \frac{1}{q(y)} \frac{dy}{dx} \, dx = \int p(x) \, dx$$

- Use the change of variables formula
  (equivalently, use the relation $dy = \dfrac{dy}{dx} dx$ for the differential dy of y in terms of dx)
  and get:

$$\int \frac{1}{q(y)} \, dy = \int p(x) dx$$

Suppose that:

- $Q(y)$ is an antiderivative of $\dfrac{1}{q(y)}$, i.e. $Q'(y) = \dfrac{1}{q(y)}$
- $P(x)$ is an antiderivative of $p(x)$, i.e. $P'(x) = p(x)$,

Then

$$\int \frac{1}{q(y)}\, dy = \int p(x) dx$$

is equivalent to

$$Q(y) = P(x) + C, \qquad C\text{=constant}$$

These are **implicit solutions**.

To find **explicit solutions** $y = \phi(x)$ one needs to solve $Q(y) = P(x) + C$ for y.
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Example:

- Determine the integral curves of the DE: $yy' = -x$.
- Solve the initial value problem $yy' = -x$, $y(0) = 1$ and determine the interval in which the solution exists.