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Remarks:

- $\frac{dy}{dt} + p(t)y = 0$ linear homogenous $\Rightarrow$ separable.
- The methods of integrating factors will allow us to solve any 1st order linear DE.

The method of integrating factors for solving first order linear DE in standard form: $\frac{dy}{dt} + p(t)y(t) = h(t)$

- Choose an antiderivative $A(t)$ of $p(t)$, i.e. $A'(t) = p(t)$.
  $A(t)$ is found by integrating $p(t)$.
- Set $\mu(t) = e^{A(t)}$: this is the **integrating factor**.
- Multiply both sides of the DE by $\mu(t) = e^{A(t)}$:

$$e^{A(t)}\frac{dy}{dt} + e^{A(t)}p(t)y = h(t)e^{A(t)}$$

- Notice that $e^{A(t)}\frac{dy}{dt} + e^{A(t)}p(t)y = \frac{d}{dt}\left(e^{A(t)}y\right)$. Hence the DE becomes

$$\frac{d}{dt}\left(e^{A(t)}y\right) = h(t)e^{A(t)}$$

- Integrate: $e^{A(t)}y = \int h(t)e^{A(t)}dt$
  i.e.

$$y = e^{-A(t)}\int h(t)e^{A(t)}dt$$

(Remark that the indefinite integral is given up to a additive constant C)

Example 1

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Example 2

Consider the first order linear DE

$$y' + 2ty = 2e^{-(t-1)^2}$$

(1) Solve the DE using the method of the integrating factor.

(2) Determine the solution safistying the initial condition $y(0) = 1$.

(3) Use the general solution to determine how solutions behave as $t \to +\infty$.