Section 2.4: Difference between linear and nonlinear ODE

In this section we are concerned with the two following questions:

Given the initial value problem (IVP):

$$\frac{dy}{dt} = f(t, y)$$
$$y(t_0) = y_0$$

- **existence**: does the IVP have a solution, and if so, where?
- **uniqueness**: is the solution unique?

We explore these questions for linear and non-linear cases.

The linear case: an example

$$\frac{dy}{dt} + \frac{1}{t-1} y = 1$$

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$$\frac{dy}{dt} + \frac{1}{t-1}y = 1$$ first order, linear, non-homogenous DE

of the form $\dfrac{dy}{dt} + p(t)y = h(t)$ with $p(t) = \dfrac{1}{t-1}$ discontinuous at $t = 1$.

- Determine the general solution of this DE.

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  Answer: $y = \frac{t^2 - 2t + C}{2(t-1)}$, where $C =$ constant. Defined for $t \neq 1$.

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- Solve the IVP:

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(1) if $(t_0, y_0) = (2, 0)$:

[*Recall:* A solution is a differentiable function $y = \phi(t)$ satisfying the IVP on some interval I containing $t = 2$. Want I as large as possible]

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Existence and uniqueness of 1st order linear IVP

Theorem (Theorem 2.4.1)

If both $p(t)$ and $g(t)$ are continuous functions of $t \in (\alpha; \beta)$ and $t_0 \in (\alpha; \beta)$, then there is a ***unique*** *solution $y = \phi(t)$ to the IVP*

$$y' + p(t)y = g(t), \quad y(t_0) = y_0$$

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Example: $(t^2 - 4)y' + ty = e^t$ with $y(1) = 1$
Determine (without solving the DE) an interval in which the solution of the IVP exists.

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  Conclusion: This IVP has three different solutions:

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The previous example shows that the continuity of $f(t, y)$ is not sufficient to guarantee the uniqueness of the solution to an IVP for a nonlinear first-order DE.

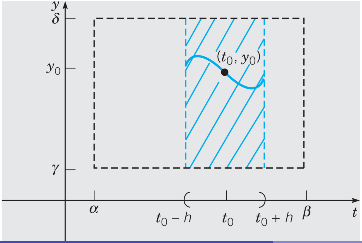
Theorem (Theorem 2.4.2)

Let $R = \{(t, y) : \alpha < t < \beta,\ \gamma < y < \delta\}$ be an open rectangle in the ty-plane and let $(t_0, y_0) \in R$.

*If both f and $\dfrac{\partial f}{\partial y}$ are continuous in $(t, y) \in R$, then there **exists** a **unique** function $y = \phi(t)$ the IVP:*

$$y' = f(t, y), \quad y(t_0) = y_0$$

*for t in **some** interval $(t_0 - h, t_0 + h)$ contained in (α, β)*



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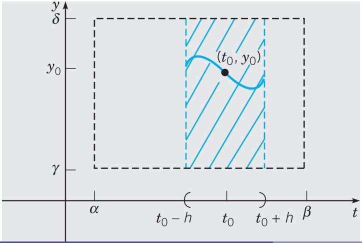

Example:
Does $y' = \frac{3}{2}y^{1/3}$, $y(0) = 0$
satisfy the conditions of this theorem?

Linear vs non-linear case

The 1st order **linear** DE $y' + p(t)y = g(t)$ has the following properties:

(1) If the p and g are continuous, there is a general solution (containing an arbitrary constant) that represents all solutions of the DE.

(2) The general solution (and hence every particular solution) has an explicit expression.

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A **nonlinear** first order ODE does not necessarily have any of the above properties.