Section 2.6: Exact differential equations

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Thus: the solution to the original DE is: $x^2y^2 + 2xy = C$, where C is a constant.

Recall that $\dfrac{\partial^2\psi}{\partial x\partial y} = \dfrac{\partial^2\psi}{\partial y\partial x}$. If $\dfrac{\partial\psi}{\partial x} = M$ and $\dfrac{\partial\psi}{\partial y} = N$, then

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Definition

The first order DE: $M(x, y) + N(x, y)\frac{dy}{dx} = 0$ is **exact** if

$$\frac{\partial M(x, y)}{\partial y} = \frac{\partial N(x, y)}{\partial x}$$

This is equivalent to the fact that there is a function $\psi(x, y)$ such that

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How to find ψ?

To solve the first order *exact* DE $M(x,y) + N(x,y)\dfrac{dy}{dx} = 0$

- **Determine** $\psi(x,y)$ so that $\dfrac{\partial\psi}{\partial x} = M$ and $\dfrac{\partial\psi}{\partial y} = N$:

(a) $\dfrac{\partial\psi}{\partial x} = M$ means (by integrating with respect to x):

$$\psi(x,y) = \int M(x,y)\,dx + h(y)$$

(b) differentiate this formula for $\psi(x,y)$ with respect to y and use that $\dfrac{\partial\psi}{\partial y} = N(x,y)$:

$$\frac{\partial\psi}{\partial y} = \frac{\partial}{\partial y}\int M(x,y)\,dx + h'(y)$$

i.e.

$$h'(y) = N(x,y) - \frac{\partial}{\partial y}\int M(x,y)\,dx$$

Rem: the RHS is in fact a function of y only because the DE is exact, i.e. $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

(c) Integrate to get $h(y)$.

- The **solution** of the DE is $\psi(x, y(x)) = C$, where C is the constant of integration.

Example: Solve the IVP: $y - 2x + (x-y)\,y' = 0, \quad y(0) = 0.$