Chapter 3: Systems of two first order DE

Section 3.1: Systems of two linear algebraic equations

Main topics:

- system of two linear equations
- matrix, determinant, trace and inverse
- solve systems with matrices
- eigenvalues and eigenvectors.

Systems of two linear equations

A **system of two linear equations** is of the form:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{cases}$$

where

- $a_{11}, a_{12}, a_{21}, a_{22}, b_1, b_2$ are fixed real numbers (the **coefficients** of the systems)
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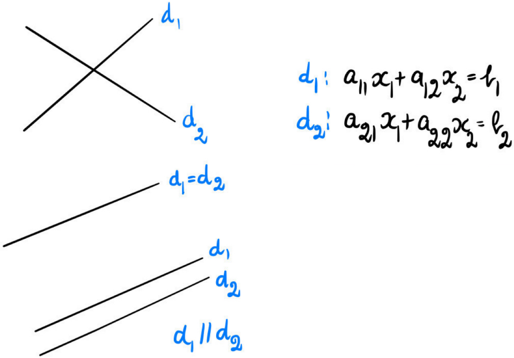
Such a system either admits

- a unique solution,
- infinitely many solutions,
- no solution at all.



Matrix notation:

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Definition

$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ is the **matrix of coefficients** of the system

$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ is column vector of the unknowns

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Definition

The system is said to be **homogeneous** if $b_1 = b_2 = 0$.

In matrix notation: $\mathbf{A}\mathbf{x} = \mathbf{0}$ where on the RHS "**0**" means the zero vector $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Determinant and trace of a 2×2 matrix

Definition

The **determinant** of $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, denoted by $\det(\mathbf{A})$, is the real number def. by

$$\det(\mathbf{A}) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}\,.$$

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Definition

$\mathbf{I} := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the **identity matrix**.

We have $\det(\mathbf{I}) = 1$ and $\operatorname{trace}(\mathbf{I}) = 2$. Moreover: $\mathbf{AI} = \mathbf{IA} = \mathbf{A}$ for any 2×2 matrix **A**.

Invertible matrices

Definition

We say that the matrix **A** is **invertible** or **non-singular** if $\det(\mathbf{A}) \neq 0$.

[It is **noninvertible** or **singular** if $\det(\mathbf{A}) = 0$.]

If the matrix **A** is invertible, then the **inverse** $\mathbf{A}^{-1}$ of **A** is the matrix uniquely defined by the formula:

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

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- The determinant of $\mathbf{B} = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}$ is $\det(\mathbf{B}) = 0$. Hence $\mathbf{B}$ is noninvertible (or singular).

Theorem

The linear system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{cases} \qquad \textit{i.e. } \mathbf{A}\mathbf{x} = \mathbf{b}$$

admits a unique solution if and only if its associated matrix $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ *is invertible.*

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Indeed: $\mathbf{A}\mathbf{x} = \mathbf{b} \Leftrightarrow \mathbf{x} = \mathbf{I}\mathbf{x} = (\mathbf{A}^{-1}\mathbf{A})\mathbf{x} = \mathbf{A}^{-1}(\mathbf{A}\mathbf{x}) = \mathbf{A}^{-1}\mathbf{b}$.

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  Hence: the system has infinitely many solutions.

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  $\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 2 & -1 \end{pmatrix}$, $\det(\mathbf{A}) = 0$ and incompatible equations.
  Hence: the system has no solution.

Eigenvalues and eigenvectors

Consider a matrix $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$.

Definition

A (real or complex) number λ is said to be an **eigenvalue** of **A** if there exists a non-zero vector $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ in $\mathbb{C}^2$ such that

$$\mathbf{A}\mathbf{v} = \lambda \mathbf{v} .$$

In this case, **v** is called an **eigenvector** of **A** corresponding to the eigenvalue λ.

If λ is a real number we say that the eigenvalue is **real**.

How to find eigenvalues?

If λ is an eigenvalue of $\mathbf{A}$ and $\mathbf{v}(\neq 0)$ a corresponding eigenvector then one has:

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v} \Longleftrightarrow (\mathbf{A} - \lambda I)\mathbf{v} = 0$$

i.e. $(\mathbf{A} - \lambda I)\mathbf{v} = 0$ has a solution $\mathbf{v}$ which is a nonzero vector

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Thus:
The eigenvalues of **A** are the numbers λ which are solutions of the equation

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0\,,$$ called the **characteristic equation** of **A**.

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$$\mathbf{Av} = \lambda \mathbf{v} \Longleftrightarrow (\mathbf{A} - \lambda I)\mathbf{v} = 0$$

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i.e. $$\det(\mathbf{A} - \lambda \mathbf{I}) = 0.$$

Thus:
The eigenvalues of **A** are the numbers λ which are solutions of the equation

$\det(\mathbf{A} - \lambda \mathbf{I}) = 0\,,$ called the **characteristic equation** of **A**.

Remark: we can always solve this equation because

$$\begin{aligned} \det(\mathbf{A} - \lambda \mathbf{I}) &= \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} \\ &= (a_{11} - \lambda)(a_{22} - \lambda) - a_{21}a_{12} \\ &= \lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{21}a_{12}) \\ &= \lambda^2 - \text{trace}(\mathbf{A})\lambda + \det(\mathbf{A}) \end{aligned}$$

is a polynomial of degree 2, called **characteristic polynomial** of **A**

Conclusion:

The eigenvalues of $\mathbf{A}$ are the **roots** of the **characteristic polynomial** $\det(\mathbf{A} - \lambda \mathbf{I})$ of $\mathbf{A}$, that is, the solutions the **characteristic equation** of $\mathbf{A}$:

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Examples

- $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ has two real eigenvalues
- $\mathbf{B} = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}$ has two complex conjugate eigenvalues.

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Remarks:

- Since **A** is a real matrix, its characteristic equation is a degree 2 equation with real coefficients (which are 1, $-\mathrm{trace}(\mathbf{A})$ and $\det(\mathbf{A})$).
- Consider the equation $\lambda^2 + b\lambda + c = 0$ where $b, c \in \mathbb{R}$. Its solutions λ_1, λ_2 are either both real numbers, or complex conjugate numbers.

How to find eigenvectors ?

- If $\mathbf{v}$ is an eigenvector for $\mathbf{A}$ for the eigenvalue λ, then $\mathbf{Av} = \lambda\mathbf{v}$, that is $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = 0$.
- Solve the system of two linear algebraic equations $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = 0$ where the coordinates of $\mathbf{v}$ are the unknowns.

Example $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$