Section 3.2: Systems of two first-order linear DE

Main topics:

- System of two first order linear differential equations
- Initial value problems
- Matrix notation
- Applications to second order differential equations.

Definition

A **system of two first order linear differential equations** has the form:

$$\begin{cases} \dfrac{dx}{dt} = p_{11}(t)x + p_{12}(t)y + g_1(t) \\ \dfrac{dy}{dt} = p_{21}(t)x + p_{22}(t)y + g_2(t) \end{cases} \tag{S}$$

where

- $p_{11}, p_{12}, p_{21}, p_{22}$ and g_1, g_2 are given functions of t, defined on a same open interval I
- $x = x(t)$, $y = y(t)$ are **two** unknown functions of t (the **state variables**).

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Definition

A **solution of (S)** consists of two differentiable functions $x = x(t)$, $y = y(t)$ satisfying (S) in some interval $I_0 \subseteq I$.

Example Check that $x(t) = -2e^{-3t}$, $y(t) = e^{-3t}$ is a solution of the first system. Which I_0?

IVP for systems of two linear DE's

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and *two* initial conditions $x(t_0) = x_0$ and $y(t_0) = y_0$ form an **initial value problem (IVP)**.

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with initial conditions $x(1) = 0$, $y(1) = 2$.

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Theorem (Theorem 3.2.1)

Suppose that the functions $p_{11}, p_{12}, p_{21}, p_{22}, g_1, g_2$ are continuous on an open interval I containing the initial value t_0. Then the IVP of the above definition has a unique solution $x = x(t), y = y(t)$ defined for all t in I.

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What is the largest interval I on which the solution of the IVP of the example exists and is unique?

System notation:

$$\begin{cases} \dfrac{dx}{dt} = p_{11}(t)x + p_{12}(t)y + g_1(t) \\ \dfrac{dy}{dt} = p_{21}(t)x + p_{22}(t)y + g_2(t) \end{cases}$$

Matrix notation:

$$\Leftrightarrow \quad \mathbf{x}'(t) = \mathbf{P}(t)\mathbf{x}(t) + \mathbf{g}(t)$$

Initial conditions:

$x(t_0) = x_0, y(t_0) = y_0 \quad \Leftrightarrow \quad \mathbf{x}(t_0) = \mathbf{x}_0$

where:

- $\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ = vector of unknown functions (the **state vector**).
- $\mathbf{x}'(t) = \dfrac{d\mathbf{x}}{dt} = \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix}$, $\underbrace{\mathbf{P}(t) = \begin{pmatrix} p_{11}(t) & p_{12}(t) \\ p_{21}(t) & p_{22}(t) \end{pmatrix}}_{\textbf{matrix of coefficients}}$, $\underbrace{\mathbf{g}(t) = \begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix}}$ the nonhomogeneous term, or **input** or **forcing function**
- $\mathbf{x}_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ = vector of initial conditions.

Definition

The system is called **homogeneous** if $\mathbf{g}(t) = 0$ for all t (i.e. $g_1(t) = g_2(t) = 0$ for all t). Otherwise, it is called **non-homogeneous**

Examples

- The matrix notation of

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$$\mathbf{P}(t) = \begin{pmatrix} t & t^2 \\ 1 & e^t \end{pmatrix} \quad \text{and} \quad \mathbf{g}(t) = \begin{pmatrix} -2 \\ -t^2 \end{pmatrix}$$

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with initial condition $x(1) = 1$ and $y(1) = 0$.

Important example: constant coefficients

Definition

A system of two linear differential equations $\mathbf{x}'(t) = \mathbf{P}(t)\mathbf{x}(t) + \mathbf{g}(t)$ is said to be **with constant coefficients** if $\mathbf{P}$ and $\mathbf{g}$ are constant in t, i.e. it is of the form

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x} + \mathbf{b}$$

where:

- $\mathbf{A}$ is a 2×2 matrix with real coefficients
- $\mathbf{b}$ is a 2×1 column vector with real coefficients.

Example

$$\begin{cases} \dfrac{dx}{dt} = 2x + y - 2 \\ \dfrac{dy}{dt} = 3x + 4y + 5 \end{cases}$$

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$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$$

Applications to 2nd order DE

Consider the second order DE:

$$y'' + p(t)y' + q(t)y = g(t)$$

with initial conditions $y(t_0) = y_0$ and $y'(t_0) = y_1$.

- Set $x_1 = y$ and $x_2 = y'$. Then $x_1' = y' = x_2$ and $x_2' = y''$.
- The DE can now be rewritten as:

$$\begin{cases} x_2' + p(t)x_2 + q(t)x_1 = g(t) \\ x_1' = x_2 \end{cases}$$

  i.e.

$$\begin{cases} x_1' = x_2 \\ x_2' = -q(t)x_1 - p(t)x_2 + g(t) \end{cases}$$

  with initial conditions $x_1(t_0) = y_0$ and $x_2(t_0) = y_1$.
- Equivently, as the system of two first order DE's

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -q(t) & -p(t) \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 \\ g(t) \end{pmatrix} \quad \text{with initial conditions } \mathbf{x}(t_0) = \begin{pmatrix} y_0 \\ y_1 \end{pmatrix}$$

Example: Transform the following DE into a system of first order equations:

$$y'' + 3ty' + 5y = t^2 + 4$$