Section 3.3: Homogenous linear systems with constant coefficients

Main topics:

- Principle of superposition
- Wronskian and linear independence of solutions
- The general solution
- Use of eigenvalues and eigenvectors

The principle of superposition

Theorem (Theorem 3.3.1)

Suppose that $\mathbf{x_1} = \mathbf{x_1}(t)$ *and* $\mathbf{x_2} = \mathbf{x_2}(t)$ *are solutions of the homog. system* $\mathbf{x}' = \mathbf{A}\mathbf{x}$*. Then, for any constants* c_1, c_2*,*

$$\mathbf{x}(t) = c_1\mathbf{x_1}(t) + c_2\mathbf{x_2}(t)$$

is also a solution of the system.

This theorem provides a tool to generate solutions from two fixed solutions.
It is known as the **principle of superposition**.

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Definition

If $\mathbf{x}(t) = c_1\mathbf{x_1}(t) + c_2\mathbf{x_2}(t)$ for all t, then we say that $\mathbf{x}$ is a **linear combination** of $\mathbf{x_1}$ and $\mathbf{x_2}$, and we write it as: $\mathbf{x} = c_1\mathbf{x_1} + c_2\mathbf{x_2}$.

Wronskian of two solutions

Let $\mathbf{x_1}(t) = \begin{pmatrix} x_{11}(t) \\ x_{21}(t) \end{pmatrix}$ and $\mathbf{x_2}(t) = \begin{pmatrix} x_{12}(t) \\ x_{22}(t) \end{pmatrix}$ be two solutions of the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

Definition

The **Wronskian** of $\mathbf{x_1}$ and $\mathbf{x_2}$ is the function $W[\mathbf{x_1}, \mathbf{x_2}]$ defined at t by the determinant

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$$W[\mathbf{x_1}, \mathbf{x_2}](t) = \begin{vmatrix} 4 & -2e^{-3t} \\ 1 & e^{-3t} \end{vmatrix} = e^{-3t}\begin{vmatrix} 4 & -2 \\ 1 & 1 \end{vmatrix} = e^{-3t}(4+2) = 6e^{-3t}.$$

Linearly independent solutions

Definition

Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be solutions of the homogenous system of linear DE's $\mathbf{x}' = A\mathbf{x}$ both defined on an interval I.

- The solutions of the system are said **linearly dependent** in an open interval I if there are constants c_1, c_2 (not both zero and *independent of t*) such that

$$c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) = 0 \text{ for all } t \in I.$$

- Two solutions that are not linearly dependent are called **linearly independent**.
- $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent if and only if $W[\mathbf{x}_1, \mathbf{x}_2](t) \neq 0$ for all t in I.
- Two solutions $\mathbf{x}_1$ and $\mathbf{x}_2$ that are linearly independent are said to form a **fundamental set of solutions**.

Remark: $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly dependent if and only if one is a constant multiple of the other: there is a constant k (independent of t) such that

$$\mathbf{x}_1(t) = k\mathbf{x}_2(t) \text{ for all } t \quad \textit{or} \quad \mathbf{x}_2(t) = k\mathbf{x}_1(t) \text{ for all } t.$$

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Are $\mathbf{x_1}$ and $\mathbf{x_2}$ linearly independent?

The general solution of $\mathbf{x}' = \mathbf{Ax}$

Theorem (Theorem 3.3.4)

Suppose $\mathbf{x_1}$ and $\mathbf{x_2}$ for two linearly independent solutions of the system $\mathbf{x}' = \mathbf{Ax}$. Then any solution of the above system is of the form

$$\mathbf{x}(t) = c_1\mathbf{x_1}(t) + c_2\mathbf{x_2}(t)$$

*for some constants c_1 and c_2. This is the **general solution** of the system.*
If, moreover, we fix an intitial condition $\mathbf{x}(t_0) = \mathbf{x}_0$, where $\mathbf{x}_0 = \begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix}$ is a constant vector, then the constants c_1 and c_2 are uniquely determined and the solution to the system is unique.

Conclusion:

- The general solution of a homogenous system of **two** linear first order DEs is a linear combination of **two linearly independent** solutions (=one is not a multiple of the other)
- To find the general solution it is enough to find two linearly independent solutions.
- An initial condition uniquely determines the constants c_1 and c_2 and hence yields a unique solution to an IVP.

Example

Let $\mathbf{A} = \begin{pmatrix} -1 & 4 \\ 1/2 & -2 \end{pmatrix}$. Consider the IVP: $\mathbf{x}' = \mathbf{A}\mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

We have shown: $\mathbf{x_1}(t) = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$ and $\mathbf{x_2}(t) = e^{-3t}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ are linearly independent solutions. (So they form a fundamental set of solutions.)

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where c_1, c_2 are arbitrary constants.

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where c_1, c_2 are arbitrary constants.

We use the initial condition to determine the value of c_1 and c_2:

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \mathbf{x}(0) = \begin{pmatrix} 4c_1 - 2c_2 e^{-3\cdot 0} \\ c_1 + c_2 e^{-3\cdot 0} \end{pmatrix} = \begin{pmatrix} 4c_1 - 2c_2 \\ c_1 + c_2 \end{pmatrix}$$

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i.e.

$$\begin{cases} 4c_1 - 2c_2 = 2 \\ c_1 + c_2 = 1 \end{cases}, \text{ that is } \begin{cases} 2c_1 - c_2 = 1 \\ c_1 + c_2 = 1 \end{cases}. \quad \text{So: } c_1 = \frac{2}{3} \text{ and } c_2 = \frac{1}{3}.$$

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Conclusion: the solution of the IVP is $\mathbf{x}(t) = \frac{2}{3}\begin{pmatrix} 4 \\ 1 \end{pmatrix} + \frac{1}{3}e^{-3t}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

How to find solutions of $\mathbf{x}' = \mathbf{A}\mathbf{x}$

The eigenvalues and eigenvectors of the matrix coefficients **A** allow us to solve the homogenous system of first order linear DE $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

Theorem

Suppose that $\mathbf{v}$ *is eigenvector of* $\mathbf{A}$ *with eingenvalue* λ*, i.e.* $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$*.*

Then $\mathbf{x}(t) = e^{t\lambda}\mathbf{v}$ *is a solution of the system* $\mathbf{x}' = \mathbf{A}\mathbf{x}$*.*

Example

$\mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector of $A = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}$ for the eigenvalue $\lambda = 2$,

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But:

To have the general solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ we need **two** linearly independent solutions.

A fact about linear independence

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We say that two vectors $\mathbf{v}_1 = \begin{pmatrix} v_{11} \\ v_{21} \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} v_{12} \\ v_{22} \end{pmatrix}$ are **linearly independent** if the determinant $\begin{vmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{vmatrix}$ is $\neq 0$.

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- Let $\mathbf{x}_1(t) = e^{t\lambda_1}\mathbf{v}_1$ and $\mathbf{x}_2(t) = e^{t\lambda_2}\mathbf{v}_2$. Then

$$W[\mathbf{x_1}, \mathbf{x_2}](t) = \begin{vmatrix} e^{t\lambda_1} v_{11} & e^{t\lambda_2} v_{12} \\ e^{t\lambda_1} v_{21} & e^{t\lambda_2} v_{22} \end{vmatrix} = e^{t(\lambda_1+\lambda_2)} \begin{vmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{vmatrix}$$

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- **Consequence:** If $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent, then $W[\mathbf{x_1}, \mathbf{x_2}](t) \neq 0$ for all t, i.e. $\mathbf{x}_1(t) = e^{t\lambda_1}\mathbf{v}_1$ and $\mathbf{x}_2(t) = e^{t\lambda_2}\mathbf{v}_2$ are linearly independent.

How to solve $\mathbf{x}' = \mathbf{A}\mathbf{x}$ (general case)

Theorem

Suppose λ_1 and λ_2 are the eigenvalues of $\mathbf{A}$. Let

$\mathbf{v}_1$ be an eigenvector of $\mathbf{A}$ of eigenvalue λ_1, i.e. $\mathbf{A}\mathbf{v}_1 = \lambda_1 \mathbf{v}_1$,
$\mathbf{v}_2$ be an eigenvector of $\mathbf{A}$ of eigenvalue λ_2, i.e. $\mathbf{A}\mathbf{v}_2 = \lambda_2 \mathbf{v}_2$.

- *If $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent, then*

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  are linearly independent solutions of $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

  In this case, the general solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is

$$\mathbf{x}(t) = c_1 e^{t\lambda_1}\mathbf{v}_1 + c_2 e^{t\lambda_2}\mathbf{v}_2$$

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  In this case, the general solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is

  $$\mathbf{x}(t) = c_1 e^{t\lambda_1}\mathbf{v}_1 + c_2 e^{t\lambda_2}\mathbf{v}_2$$

  where c_1, c_2 are arbitrary constants.

- *For instance, if $\lambda_1 \neq \lambda_2$, then $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent. So, the above holds true.*

Example:

Consider the system of two linear differential equations $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix}.$$

- Find the eigenvalues λ_1 and λ_2 of $\mathbf{A}$.
- Find the eigenvectors of $\mathbf{A}$ of eigenvalue λ_1 and those of eigenvalue λ_2.
- Fix an eigenvector $\mathbf{v}_1$ of eigenvalue λ_1 and verify that $\mathbf{x}_1(t) = e^{\lambda_1 t}\mathbf{v}_1$ is a solution of the system.
- **Some linear algebra:** can you write the trace $\mathrm{trace}(\mathbf{A})$ of $\mathbf{A}$ and the determinant $\det(\mathbf{A})$ of $\mathbf{A}$ in terms of the eigenvalues λ_1 and λ_2?
- Determine the general solution $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

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- Determine the general solution $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

  Answer:
  $$\mathbf{x}(t) = C_1 e^{-3t}\begin{pmatrix} 1 \\ -3 \end{pmatrix} + C_2 e^{-2t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}, \qquad C_1, C_2 \text{ constants}$$
- Solve the IVP: $\mathbf{x}' = \mathbf{A}\mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$

The section 3.3 is not over...
... to be continued later...