Section 3.4: Complex eigenvalues

Consider the homogeneous system $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

In this section we suppose that the matrix **A** has two complex-conjugate (and not real) eigenvalues:

$$\lambda = \mu + i\nu \quad \text{and} \quad \overline{\lambda} = \mu - i\nu$$

where μ, ν are real numbers.

In particular: λ and $\overline{\lambda}$ are distinct and non-zero.

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- Eigenvectors $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ associated with complex eigenvalues have usually complex components $v_1 = a_1 + ib_1$, $v_2 = a_2 + ib_2$ (with $a_1, b_1, a_2, b_2 \in \mathbb{R}$).
- If $\mathbf{v} = \mathbf{a} + i\mathbf{b} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + i \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ is an eigenvector of eigenvalue λ, i.e. $A\mathbf{v} = \lambda\mathbf{v}$ then $\overline{v} = \mathbf{a} - i\mathbf{b} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} - i \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ is eigenvector of eigenvalue $\overline{\lambda}$, i.e. $A\overline{\mathbf{v}} = \overline{\lambda}\,\overline{\mathbf{v}}$.

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Example:

Determine the eigenvalues and the corresponding eigenvectors for $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

We have two linearly independent **complex-valued** solutions of $\mathbf{x}' = \mathbf{A}\mathbf{x}$, namely

$$\mathbf{x}_1(t) = e^{\lambda t}\mathbf{v} = e^{(\mu+i\nu)t}\mathbf{v} \qquad \text{and} \qquad \mathbf{x}_2(t) = e^{\overline{\lambda} t}\overline{\mathbf{v}} = e^{(\mu-i\nu)t}\overline{\mathbf{v}}$$

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Conclusion: The general solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is:

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Determine the general solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ where $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

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General solution:

$$\mathbf{x}(t) = C_1 \,\mathrm{Re}\, \mathbf{x}_1(t) + C_2 \,\mathrm{Im}\, \mathbf{x}_1(t) = C_1 \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + C_1 \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$$

where C_1, C_2 are real constants.

Section 3.4 is not over...
... to be continued later...