Section 3.5: Repeated eigenvalues

We suppose that **A** is a 2×2 matrix with two (necessarily real) equal eigenvalues $\lambda_1 = \lambda_2$. To shorten the notation, **write** λ **instead of** $\lambda_1 = \lambda_2$.

A matrix **A** with two repeated eigenvalues can have:

- **two** linearly independent eigenvectors, if $\mathbf{A} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$.
- **one** linearly independent eigenvector, if $\mathbf{A} \neq \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$.

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Show that $\mathbf{A} = \begin{pmatrix} -1/2 & 0 \\ 0 & -1/2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -1/2 & 1 \\ 0 & -1/2 \end{pmatrix}$ have one repeated eigenvalue λ. Find λ.

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Show that **A** has two linearly independent eigenvectors of eigenvalue λ whereas **B** does not.

$\left[\text{For instance: } \mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ for } \mathbf{A}; \quad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ for } \mathbf{B} \right]$

Keep the above notation.

- If there are two linearly independent eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ of eigenvalue λ, i.e. if $\mathbf{A} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$:
  Then two linearly independent solutions of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ are
  $$\mathbf{x}_1(t) = e^{\lambda t}\mathbf{v}_1 \qquad \text{and} \qquad \mathbf{x}_2(t) = e^{\lambda t}\mathbf{v}_2$$
  The **general solution** is
  $$\mathbf{x}(t) = C_1\mathbf{x}_1(t) + C_2\mathbf{x}_2(t) = C_1 e^{\lambda t}\mathbf{v}_1 + C_2 e^{\lambda t}\mathbf{v}_2.$$
  (This case enters in the Theorem stated at the end of Section 3.3).
- If there is only one linearly independent eigenvector $\mathbf{v}_1$ of eigenvalue λ, i.e. if $\mathbf{A} \neq \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$:
  Then two linearly independent solutions of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ are
  $$\mathbf{x}_1(t) = e^{\lambda t}\mathbf{v}_1 \qquad \text{and} \qquad \mathbf{x}_2(t) = e^{\lambda t}(t\mathbf{v}_1 + \mathbf{w}) = t\mathbf{x}_1(t) + e^{\lambda t}\mathbf{w}.$$
  where $\mathbf{w}$ satisfies $(\mathbf{A} - \lambda\mathbf{I})\mathbf{w} = \mathbf{v}_1$
  (we say that $\mathbf{w}$ is a *generalized eigenvector* corresponding to the eigenvalue λ).
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Solution:

- $\mathbf{B}$ has one repeated eigenvalue $\lambda = -1/2$ and one linearly independent eigenvector $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. This gives one solution $\mathbf{x}_1(t) = e^{-\frac{1}{2}t}\mathbf{v}_1$.

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  Get $w_2 = 1$, i.e. $\mathbf{w} = \begin{pmatrix} w_1 \\ 1 \end{pmatrix}$, where $w_1 \in \mathbb{R}$ can be chosen as we want, e.g. $w_1 = 0$.

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  Then
  $$\mathbf{x}_2(t) = e^{\lambda t}(t\mathbf{v}_1 + \mathbf{w}) = e^{-\frac{1}{2}t}\left(t\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = e^{-\frac{1}{2}t}\begin{pmatrix} t \\ 1 \end{pmatrix}.$$

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Remark: A different choice of w_2 (for instance $w_2 = 2$) would give a different $\mathbf{x}_2$ but the **same** general solution (try).

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Section 3.5 is not over...
... to be continued later...

Summary: the general solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$

(**A** is a 2×2 matrix with real entries)

The form of the general solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ depends on the eigenvalues λ_1, λ_2 of **A** as follows:

(1) λ_1, λ_2 **both real and distinct:** $\quad \mathbf{x}(t) = C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2$

where:
- ◇ $\mathbf{v}_1$ is an eigenvector of **A** of eigenvalue λ_1,
- ◇ $\mathbf{v}_2$ is an eigenvector of **A** of eigenvalue λ_2,
- ◇ C_1, C_2 are real constants.

(2) λ_1, λ_2 **complex-conjugate & not real**: $\quad \mathbf{x}(t) = C_1 \operatorname{Re}\mathbf{x}_1(t) + C_2 \operatorname{Im}\mathbf{x}_1(t)$

where:
- ◇ $\mathbf{x}_1(t) = e^{\lambda_1 t}\mathbf{v}_1$ and $\mathbf{v}_1$ is an eigenvector of **A** of eigenvalue λ_1,
- ◇ C_1, C_2 are real constants.

(3) $\lambda_1 = \lambda_2$ **both real and** $\mathbf{A} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix}$**:** $\quad$ same form as in case (1).

(4) $\lambda_1 = \lambda_2$ **both real and** $\mathbf{A} \neq \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix}$**:** $\quad \mathbf{x}(t) = C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_1 t}(t\mathbf{v}_1 + \mathbf{w})$

where:
- ◇ $\mathbf{v}_1$ is an eigenvector of **A** of eigenvalue λ_1,
- ◇ $\mathbf{w}$ satisfies $(\mathbf{A} - \lambda_1 \mathbf{I})\mathbf{w} = \mathbf{v}_1$
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Remark: the constant C_1, C_2 are also called **free parameters**.

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Definitions, matrix notation, and examples (Section 3.2)
Applications to 2nd order linear differential equations (Section 3.2)

Special case of homogeneous systems with constant coefficients $\mathbf{x}' = \mathbf{A}\mathbf{x}$*:*

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- Wronskian and linear independence of solutions (Section 3.3)
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Geometric methods (direction fields, equilibrium solutions, phase portraits)
back to two first-order linear DE with constant coefficients $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{b}$

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