Section 3.6: A brief introduction to nonlinear systems

Mostly studied so far in Chapter 3:
systems of two first order **linear** DEs, i.e. systems of the form

$$\frac{dx}{dt} = p_{11}(t)x + p_{12}(t)y + g_1(t)$$
$$\frac{dy}{dt} = p_{21}(t)x + p_{22}(t)y + g_2(t)\,,$$

which include in particular:

- **autonomous** (=constant coefficients) systems of two first order **linear** DEs:

$$\frac{dx}{dt} = a_{11}x + a_{12}y + b_1$$
$$\frac{dy}{dt} = a_{21}x + a_{22}y + b_2\,,$$

  where $a_{11}, a_{12}, a_{21}, a_{22}, b_1, b_2 \in \mathbb{R}$;

- **autonomous homogenous** systems of two first order **linear** DEs:
  i.e. autonomous and with $b_1 = b_2 = 0$.

In this section: **nonlinear systems**.

- **Motivation:** most natural phenomena are essentially nonlinear: fluid dynamics, general relativity, chaos, power-flow study, ...
- **Difficulties:** solutions of non linear systems are complicated, hard to predict and hard to determine.

Definition

A general two-dimensional first order differential system is of the form

$$\frac{dx}{dt} = f(t, x, y)$$
$$\frac{dy}{dt} = g(t, x, y)$$

where f and g are arbitrary functions.

The system is said to be **nonlinear** if (at least) one of the functions $f(t, x, y)$ or $g(t, x, y)$ *is not* of the form $a(t)x + b(t)y + c(t)$, where a, b and c are functions of t.

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A general **initial value problem** is a general two-dimensional first order differential system together with two initial conditions

$$x(t_0) = x_0 \qquad \text{and} \qquad y(t_0) = y_0$$

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Vector notation:

$\mathbf{x}' = \mathbf{f}(t, \mathbf{x})$ for a general system and $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$.

or $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ if the system is autonomous.

Initial condition: $\mathbf{x}(t_0) = \mathbf{x}_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$.

Theorem (Theorem 3.6.1)

Consider a differential system

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} f(t,x,y) \\ g(t,x,y) \end{pmatrix}$$

Let R be the open rectangular cuboid R of the txy-space defined by

$$\alpha < t < \beta,\ \alpha_1 < x < \beta_1\ ,\alpha_2 < y < \beta_2$$

and let (t_0, x_0, y_0) *be a fixed element in R.*

Suppose that the functions f and g, as well as their partial derivatives $\partial f/\partial x$, $\partial f/\partial y$, $\partial g/\partial x$ *and* $\partial g/\partial y$ *are all continuous in R.*

Then there exists an open interval $I = (t_0 - h; t_0 + h)$ *in which there* ***exists*** *a* ***unique solution*** *of the given system with initial condition* $x(t_0) = x_0$ *and* $y(t_0) = y_0$.

General autonomous systems of two DEs

Consider an autonomous system

$$\mathbf{x}' = \mathbf{f}(\mathbf{x})$$

with $\mathbf{f}(\mathbf{x}) = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}$.

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- The **equilibrium solutions** or **critical points** are those solutions $\mathbf{x} = \mathbf{x}(t)$ such that $\mathbf{x}' = \mathbf{0}$, i.e. the solutions to $\mathbf{f}(\mathbf{x}) = \mathbf{0}$.

  In terms of component functions: the $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ such that $\begin{cases} f(x,y) = 0 \\ g(x,y) = 0 \end{cases}$

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- *Study of solutions and their trajectories in the phase plane:*

$$\frac{g(x,y)}{f(x,y)} = \frac{dy/dt}{dx/dt} = \frac{dy}{dx}$$

  can be viewed as a first order differential equation where the unknown is the function y and the independent variable is the function x.

  We can solve this 1st order non-linear autonomous DE **if** it is exact, separable, Bernoulli or homogenous...

  If solutions can be found, they are usually in implicit form $H(x,y) = C$, where C is a constant.

Example:

Consider the system:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -xy \\ x^2 \end{pmatrix}$$

- Find all the critical points.
- Find an equation of the form $H(x, y) = c$ satisfied by the solutions of the given system.
- Plot the phase portrait, level curves of H, and the trajectory that passes through the point $(x, y) = (2, 0)$.