Chapter 4: Second Order Linear Equations

4.1: Definitions and Examples

Definition

A second order differential equation is said to be **linear** if it can be written in **standard form**

$$y'' + p(t)y' + q(t)y = g(t)$$

where p, q and g are arbitrary functions of the independent variable t.

If the function is g identically zero, then the equation is said to be **homogeneous**. It is said to be **nonhomogeneous** otherwise.

The equation is said to be a **constant coefficient equation** if the functions p, q and g are constant (i.e. do not depend on t). Otherwise, it is said to have **variable coefficients**.

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Model:
vertical y-axis, positive direction downward,
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$y(t)$= position of the mass at time t
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Equilibrium position: $y(0) = 0$.

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Newton's law of motion:

$$my''(t) = F_{\text{net}}(t)$$

where:
$y''(t)$= acceleration of the mass at time t
$F_{\text{net}}(t)$= the net force acting on the mass, with different components.

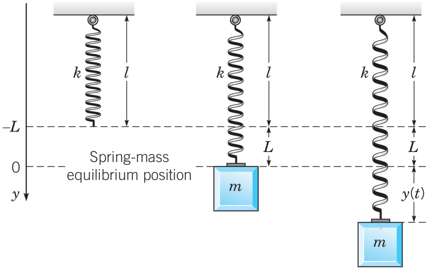

Figure 4.3.1 in: J. Brannan & W. Boyce, Differential Equations

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- **Spring force F_s:** the force exerced on the mass by the spring. It is proportional to the total elongation $L + y$ of the spring and opposite to the displacement, i.e.

$$F_s(t) = -k(L + y(t)) \qquad (\textit{Hooke's law})$$

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  At $y = 0$ the mass is in equilibrium. So the spring and the gravitational force balance each others: $mg - kL = 0$.
- **Damping force** (or **resistive force**) F_d**:** it may arise from several sources such as air resistance, friction, or additional devices. It is proportional and opposite to the velocity of the mass, i.e.

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Hence: $F_{\text{net}}(t) = W + F_s(t) + F_d(t) + F(t) = -ky(t) - \gamma y'(t) + F(t)$

With $F_{\text{net}}(t) = -ky(t) - \gamma y'(t) + F(t)$, Newton's law of motion: $my''(t) = F_{\text{net}}(t)$ becomes:

$$my''(t) + \gamma y'(t) + ky(t) = F(t)$$

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Four important special cases:

Unforced, undamped oscillator:	$my''(t) + ky(t) = 0$
Unforced, damped oscillator:	$my''(t) + \gamma y'(t) + ky(t) = 0$
Forced, undamped oscillator:	$my''(t) + ky(t) = F(t)$
Forced, damped oscillator:	$my''(t) + \gamma y'(t) + ky(t) = F(t)$

Example:

A mass weighting 2 lb stretches a spring 6 in. The mass is pulled down an additional 3 in and at time $t = 0$ it is released at a velocity of 1 ft/s in the upward direction. Assume there is no damping.

Write down the appropriate initial value problem.