Section 4.4: Vibrations. Harmonic oscillators

Main Topics:

- **Mechanical vibrations (systems spring-mass)**
- **Harmonic oscillators**
- **Examples of second order differential equations.**

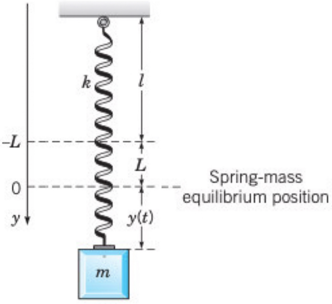
Recall the equation describing the dynamics (or the vibrations) of a spring-mass system:

$$my'' + \gamma y' + ky = F$$

where

- the unknown function $y = y(t)$ describes the motion of the mass at time t,
- m is the **mass**,
- k the **spring constant**,
- γ the **damping factor**,
- $F = F(t)$ an **external force** applied on the system.

(m, k and γ are positive constants).



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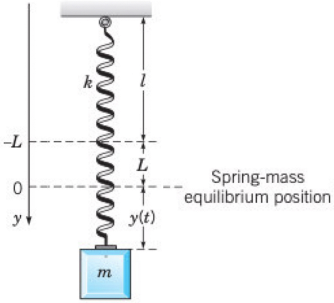

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The initial conditions:

$$y(0) = y_0 \qquad \text{and} \qquad y'(0) = v_0$$

(specifying the initial position y_0 and the initial velocity v_0)
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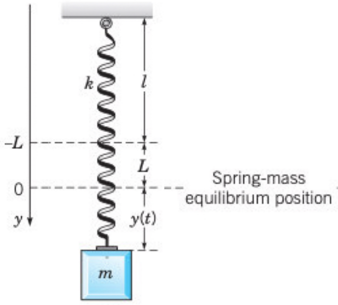

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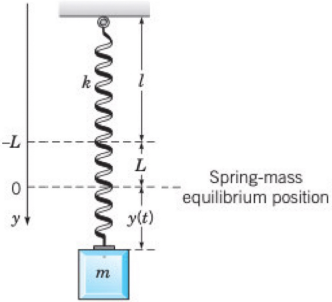

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- If $F = 0$ we say that the system is **unforced** or **free**.
- Free spring-mass systems are known as **harmonic oscillators**.

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Characteristic equation: $\lambda^2 + \omega_0^2 = 0$, with roots $\lambda = \pm i\omega_0$.

The general solution is:

$$y(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$$

for arbitrary constants A and B.

Rewrite $y(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$ by setting

$$\begin{cases} R = \sqrt{A^2 + B^2} \\ A = R\cos\delta \\ B = R\sin\delta \end{cases}$$

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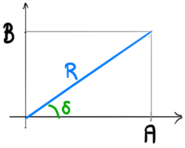

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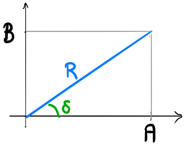

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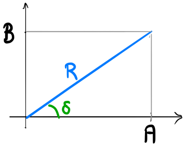

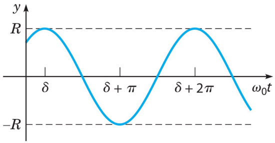

Simple harmonic motion $y = R\cos(\omega_0 t - \delta)$.

Figure 4.4.3 in J. Brannan & W. Joyce

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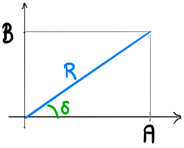

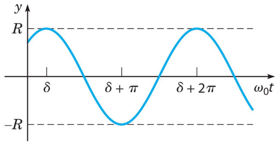

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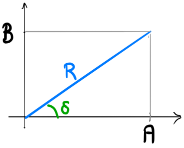

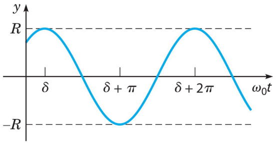

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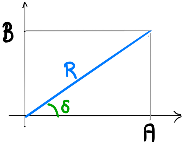

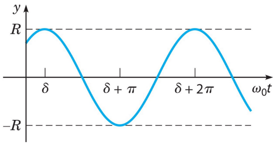

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The graph of $y(t) = R\cos(\omega_0 t - \delta)$ is a displaced cosine function.

This is a **periodic motion** (or **simple harmonic motion**) of the mass:

- $T = \frac{2\pi}{\omega_0}$ is the **period**,
- $\omega_0 = \sqrt{k/m}$ is the **natural frequency**,
- δ is the **phase**,
- R is the **amplitude**.

Example:

Determine ω_0 (the natural frequency), R (the amplitude) and δ (the phase) such that $y = -\cos t + \sqrt{3}\sin t$ can be written as $y = R\cos(\omega_0 t - \delta)$.

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Solution:

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Characteristic equation: $m\lambda^2 + \gamma\lambda + k = 0$ with roots:

$$\lambda_1, \lambda_2 = \frac{-\gamma \pm \sqrt{\gamma^2 - 4km}}{2m} = \frac{\gamma}{2m}\Big(-1 \pm \sqrt{1 - \frac{4km}{\gamma^2}}\Big)$$

The solutions of the equation of the motion are of three different types, depending on the sign of $\gamma^2 - 4km$.

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- **Underdamped harmonic motion:** $\gamma^2 - 4km < 0$
  Complex conjugate roots $\lambda_1 = \mu + i\nu$, $\lambda_2 = \overline{\lambda_1}$.
  General solution: $\quad y(t) = e^{-\gamma t/2m}(A\cos(\nu t) + B\sin(\nu t))$
  with $\mu = -\gamma/2m < 0$ and $\nu = \dfrac{\sqrt{4km - \gamma^2}}{2m} > 0$.
- **Critically damped harmonic motion:** $\gamma^2 - 4km = 0$
  Repeated *negative* roots $\lambda_1 = \lambda_2 = -\gamma/2m < 0$.
  General solution: $\quad y(t) = (A + Bt)e^{-\gamma t/2m}$
- **Overdamped harmonic motion:** $\gamma^2 - 4km > 0$
  Two distinct *negative* eigenvalues λ_1, λ_2 (because $\sqrt{\gamma^2 - 4km} < \gamma$).
  General solution: $\quad y(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$

$$\lambda_1, \lambda_2 = \frac{-\gamma \pm \sqrt{\gamma^2 - 4km}}{2m} = \frac{\gamma}{2m}\Big(-1 \pm \sqrt{1 - \frac{4km}{\gamma^2}}\Big)$$

- **Underdamped harmonic motion:** $\gamma^2 - 4km < 0$

  Complex conjugate roots $\lambda_1 = \mu + i\nu$, $\lambda_2 = \overline{\lambda_1}$.

  General solution: $y(t) = e^{-\gamma t/2m}(A\cos(\nu t) + B\sin(\nu t))$

  with $\mu = -\gamma/2m < 0$ and $\nu = \dfrac{\sqrt{4km - \gamma^2}}{2m} > 0$.

- **Critically damped harmonic motion:** $\gamma^2 - 4km = 0$

  Repeated *negative* roots $\lambda_1 = \lambda_2 = -\gamma/2m < 0$.

  General solution: $y(t) = (A + Bt)e^{-\gamma t/2m}$

- **Overdamped harmonic motion:** $\gamma^2 - 4km > 0$

  Two distinct *negative* eigenvalues λ_1, λ_2 (because $\sqrt{\gamma^2 - 4km} < \gamma$).

  General solution: $y(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$

Remark: In all cases $\lim_{t\to+\infty} y(t) = 0$
(because of the damping, as t increases, the motion decreases and eventually stops)

A more detailed study of the underdamped harmonic motion: $\gamma^2 - 4km < 0$

General solution: $y(t) = e^{-\gamma t/2m}(A\cos(\nu t) + B\sin(\nu t))$

with $-\gamma/2m < 0$ and

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$$\begin{cases} R = \sqrt{A^2 + B^2} \\ A = R\cos\delta \\ B = R\sin\delta \end{cases}$$

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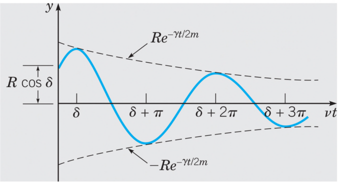

Damped vibration; $y = Re^{-\gamma t/2m}\cos(\nu t - \delta)$.

Figure 4.4.5 in J.Brannan & W. Joyce

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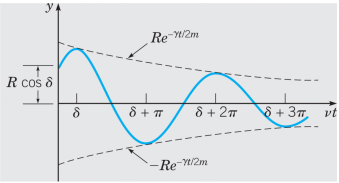

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- the motion oscillates between the curves $y = Re^{-\gamma t/2m}$ (damped oscillation; decreasing amplitude),
- ν is the **quasi-frequency**,
- $T_d = \frac{2\pi}{\nu}$ is the **quasi-period**.
- As $\gamma \to 2\sqrt{km}$ we have $\nu \to 0$ and $T_d \to \infty$ and there is no oscillation (critically damped motion). The value $\gamma = 2\sqrt{km}$ is called the **critical damping**.

Example:

An object of mass $m = 0.2$ kg is hung from a spring with spring constant $k = 40$ N/m. The object is subject to a damping with damping coefficient $\gamma = 4$ Ns/m. Suppose that there is no external force acting on the spring-mass system.

- Set up the differential equation of the motion.
- Classify the type of harmonic oscillator.
- Determine the general solution of the differential equation you found in part (a).
- Suppose that at time $t = 0$ the mass is pulled down 0.5 m below its equilibrium position and then released (i.e. the initial velocity is 0).

  Determine the motion $y(t)$ of the mass as a function of the time t.
- Write $y(t)$ in the form $y(t) = h(t)\cos(\nu t - \delta)$ for a suitable function $h(t)$ and suitable constants ν and δ?
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- Set up the differential equation of the motion.

  $my'' + \gamma y' + ky = 0$, i.e. $0.2y'' + 4y' + 40y = 0$ i.e. $y'' + 20y' + 200y = 0$, where $y = y(t)$ is the position at time t of the mass along the downward pointing vertical y-axis.

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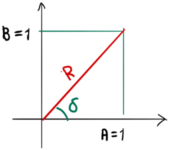

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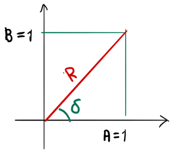

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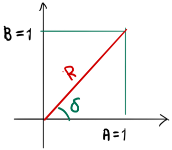

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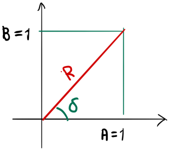

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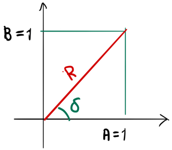

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$y(t)$ oscillates with quasi-period $T = 2\pi/\nu = \pi/5$ and decreasing "amplitude" $h(t) = \frac{\sqrt{2}}{2}e^{-10t}$.
We have $\lim_{t\to+\infty} y(t) = 0$ because $\lim_{t\to+\infty} h(t) = 0$ and $\cos(10t - \pi/4)$ stays bounded between $[-1, 1]$.

Phase portraits for harmonic oscillators

Phase portraits are obtained from corresponding systems of first order differential equations:

$$\mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{pmatrix} 0 & 1 \\ -k/m & -\gamma/m \end{pmatrix} \mathbf{x}$$

The origin is the unique equilibrium point because A is invertible (as $\det(A) = k/m \neq 0$).

The roots of the characteristic equation $m\lambda^2 + \gamma\lambda + k = 0$ are the eigenvalues of **A**.

They give the nature of the equilibrium point of the different harmonic oscillators:

- a **stable center** for the undamped harmonic oscillator,
- a **spiral sink** for the underdamped harmonic oscillator,
- a **stable improper node** for the the critically damped harmonic oscillator.
- a **nodal sink** for both the overdamped oscillators.

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Example:
For the DE $y'' + 20y' + 200y = 0$ considered in the previous example (underdamped harmonic oscillator), the origin is the unique equilibrium solution, and it is a spiral sink.