Section 5.1: Definition of the Laplace transform

Main Topics:

- **Laplace transform of a function**
- **Piecewise continuity**
- **Examples**
- **Existence theorems**

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Pierre-Simon, marquis de Laplace (1745-1827).

Portrait de Paulin Guérin, château de Versailles.

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Improper integrals

Definition

Suppose $a \in \mathbb{R}$ and f is a function defined on the interval $[a, +\infty)$.

The **improper integral** of f from a to $+\infty$, denoted $\int_a^{+\infty} f(t)\,dt$, is defined as the limit

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- Otherwise, we say that the improper integral **diverges**.

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Conclusion: the improper integral converges to the value $\frac{1}{s}$.

The Laplace transform

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Let f be a function defined on $[0, +\infty)$. The **Laplace transform of** f is the function F defined by

$$F(s) = \int_0^{+\infty} e^{-st} f(t)\, dt$$

for all values s for which this improper integral converges.

The Laplace transform of f will also be denoted by $\mathcal{L}\{f\}$.

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- Departing from the usual functional notation, one often writes "$F(s) = \mathcal{L}\{f(t)\}$".
  This means: the function F (which is function of s) is the Laplace transform of f (which is a function of t).
  The proper (but too long to write) notation would be: "given $f = f(t)$, consider $F(s) = \mathcal{L}\{f\}(s)$."

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- For $a = 0$, we deduce the Laplace transform of the constant function $f(t) = 1$ for all $t \geq 0$:

$$\mathcal{L}\{1\} = \frac{1}{s}\,, \qquad s > 0\,.$$

Examples (continued):

- If we replace $a \in \mathbb{R}$ with $a + ib \in \mathbb{C}$ (where $a, b \in \mathbb{R}$), then

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Remark: This is often written on a table omitting the “s” variable on the RHS:

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Theorem (Theorem 5.1.2)

Suppose that:

- f_1 *is a function whose Laplace transform exists on the interval* $(a_1, +\infty)$*,*
- f_2 *is a function whose Laplace transform exists on the interval* $(a_2, +\infty)$*.*

Then, for any (real or complex) constants c_1*,* c_2*, the Laplace transform of* $c_1 f_1 + c_2 f_2$ *exists on the interval* $(\max\{a_1, a_2\}, +\infty)$*, and satisfies*

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Example: Find the Laplace transform of $f(t) = \sin(bt)$.

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Definition (Definition 5.1.3)

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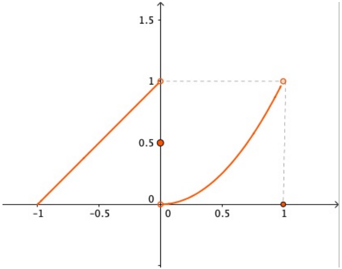

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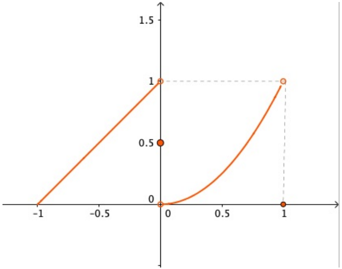

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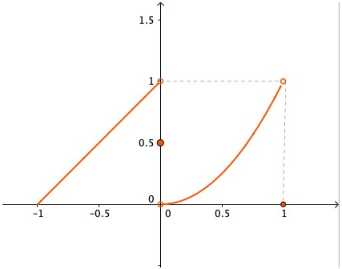

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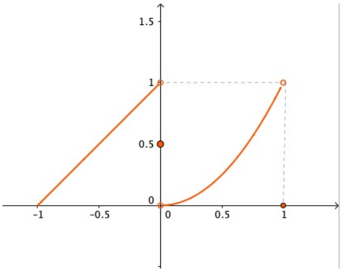

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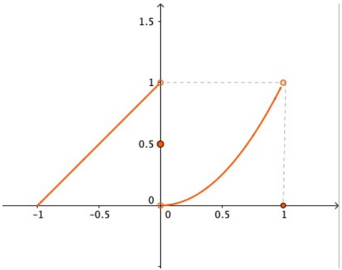

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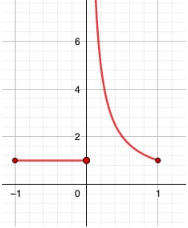

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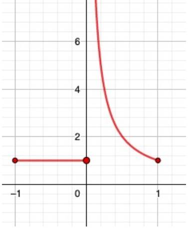

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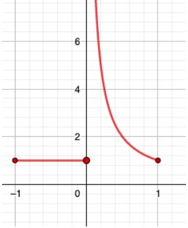

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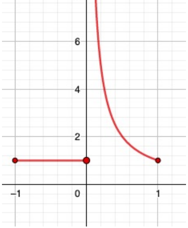

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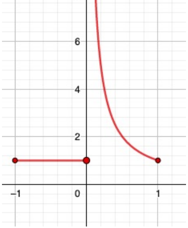

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Partition of $(-\infty, \infty)$ by $-\infty < 0 < +\infty$
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- *If $f(t) \geq g(t) \geq 0$ for $t \geq M$, and if $\int_M^{+\infty} g(t)\,dt$ diverges then $\int_a^{+\infty} f(t)\,dt$ also diverges.*

Functions of exponential order

Definition (Definition 5.1.5)

A function f is of **exponential order** (as $t \to +\infty$) if there exist real constants $M \geq 0$, $K > 0$ and a such that

$$|f(t)| \leq Ke^{at} \qquad \text{for } t \geq M\,.$$

Remarks:

- The choice of constants M, K, a is not unique.
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Existence of Laplace transforms

Theorem (Theorem 5.1.6, Corollary 5.1.7)

Suppose:

- f *is piecewise continuous on* $[0, A]$ *for any positive real number* A
- f *is of exponential order, that is* $|f(t)| \leq Ke^{at}$ *for* $t \geq M$.

Then:

(1) *the Laplace transform of* f *exists for* $s > a$,

(2) *there exists a positive constant* L *such that*

$$|\mathcal{L}\{f\}(s)| \leq L/s \qquad \text{for all } s \text{ sufficiently large.}$$

In particular: $\lim_{s\to+\infty} \mathcal{L}\{f\}(s) = 0$.