Section 5.2: Properties of the Laplace transform

Main Topics:

- Laplace transform of $e^{ct}f$,
- Laplace transform of derivatives
- Laplace transform of $t^n f$.
- Laplace transform of differential equations.

Recall that the Laplace transform of a function *f* is defined by

$$\mathcal{L}{f}(s) = \int_0^{+\infty} e^{-st} f(t), dt$$

for all values $s \in \mathbb{R}$ for which this integral converges.

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Theorem (Theorem 5.2.1)

If the Laplace transform of f exists for s > a, then one has

$$\mathcal{L}\lbrace e^{ct}f\rbrace(s) = \mathcal{L}\lbrace f\rbrace(s-c) \qquad \text{for } s>a+c$$

for a constant c.

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$$\mathcal{L}\{g\}(s) = \int_0^{+\infty} e^{-st} e^{-3t} \sin(2t) dt = \int_0^{+\infty} e^{-(s+3)t} \sin(2t) dt$$



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$$= \frac{2}{(s+3)^2+4} \quad \text{for } s+3>0, \text{ i.e. } s>-3.$$

Laplace Transform of derivatives

Recall (Definition 5.1.5) that g is of exponential order if there exist real constants $M \ge 0$, K > 0 and a such that $|g(t)| \le Ke^{at}$ for $t \ge M$.

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Suppose that:

- f is continuous on the interval $0 \le t \le A$ for all A,
- f' is piecewise continuous on the interval $0 \le t \le A$ for all A,
- f and f' are of exponential order with exponent a.

Then the Laplace transform of f' exists for s > a and is given by

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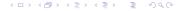
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Example:

- Compute the Laplace transform of $g(t)=\cos(bt)$ where $b\in\mathbb{R},\ b\neq 0$. [Recall that $\mathcal{L}\{e^{\pm ibt}\}(s)=\frac{1}{s\mp ib}$ for s>0.]
- Verify the formula $\mathcal{L}\{f'\}(s) = s\mathcal{L}\{f\}(s) f(0)$ for $f(t) = \sin(bt)$.



$$\mathcal{L}\{\cos(bt)\}(s) = \mathcal{L}\left\{\frac{e^{ibt} + e^{-ibt}}{2}\right\}(s)$$

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On the other hand,

$$s\mathcal{L}{f}(s) - f(0) = s\frac{b}{s^2 + b^2} - 0$$
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The formula is therefore verified.



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Example:

If f', f'' satisfy the same conditions of f, f' in Theorem 5.2.2, then for s > a

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Corollary (Corollary 5.2.3)

Suppose that

- $f, f', \ldots, f^{(n-1)}$ are continuous on any interval $0 \le t \le A$,
- $f^{(n)}$ is piecewise continuous on any interval $0 \le t \le A$,
- f, f', ..., f⁽ⁿ⁾ are of exponential order, with exponent a.

Then the Laplace transform of $f^{(n)}$ exists for s > a and is given by

$$\mathcal{L}\{f^{(n)}\}(s) = s^n \mathcal{L}\{f\}(s) - s^{(n-1)}f(0) - \dots - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

Theorem (Theorem 5.2.4)

Suppose

- f is piecewise continuous on any interval $0 \le t \le A$,
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Then for any positive integer n, we have

$$\mathcal{L}\{t^n f\}(s) = (-1)^n (\mathcal{L}\{f\})^{(n)}(s)$$
 for $s > a$.

Example:

Compute $\mathcal{L}\{t\}$, $\mathcal{L}\{t^2\}$ and $\mathcal{L}\{2t^2-3t+1\}$.

[Recall that
$$\mathcal{L}\{1\}(s) = \frac{1}{s}$$
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Therefore:

$$\mathcal{L}\{2t^2 - 3t + 1\} = 2\mathcal{L}\{t^2\} - 3\mathcal{L}\{t\} + \mathcal{L}\{1\} = \frac{4}{s^3} + 3\frac{1}{s^2} + \frac{1}{s} \quad \text{for } s > 0.$$



Ву

$$\left(\frac{1}{s}\right)^{(n)} = (-1)^n \frac{n!}{s^{n+1}},$$

we get the following corollary:

Corollary (Corollary 5.2.5)

For any positive integer n, we have

$$\mathcal{L}\lbrace t^n \rbrace (s) = \frac{n!}{s^{n+1}} \quad \text{for } s > 0.$$

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$$\mathcal{L}\{y''\} - 3\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{e^{-3t}\}$$

Recall that $\mathcal{L}\lbrace e^{-3t}\rbrace = \frac{1}{s+3}$ for s > -3.

Set $Y = \mathcal{L}\{y\}$. We assume that our unknown solution y satisfies all the the hypothesis of Corollary 5.2.3. This allows to compute $\mathcal{L}\{y'\}$ and $\mathcal{L}\{y''\}$ in (*):

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$$(s^2 - 3s + 2)Y(s) - s + 3 = \frac{1}{s+3}$$

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The problem of finding a function out its Laplace transform is the problem of inverting the Laplace transform.