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Main Topics:

- **Inverse Laplace transform**
- **Tables of Laplace transforms**
- **Integrals of partial fractions**

Inverse Laplace transform of a piecewise continuous function

Theorem (Existence of the inverse Laplace transform)

Suppose that $f(t)$ *and* $g(t)$ *are piecewise continuous and of exponential order on* $[0, +\infty)$.

If $\mathcal{L}\{f\} = \mathcal{L}\{g\}$, *then* $f = g$ *at all points where* f *and* g *are continuous.*

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If $\mathcal{L}\{f\} = \mathcal{L}\{g\}$, then $f = g$ at all points where f and g are continuous.

This theorem allows us to define in an essentially unique way the inverse Laplace transform.

Definition (Definition 5.3.2)

Let f be a piecewise continuous and of exponential order on $[0, +\infty)$.
If $F = \mathcal{L}\{f\}$, then f is called **inverse Laplace transform** of F, and we write it $f = \mathcal{L}^{-1}(F)$.

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From the linearity of the Laplace transform, we obtain the linearity of its inverse:

Theorem (Theorem 5.3.3)

Suppose that $f_1 = \mathcal{L}^{-1}(F_1)$ *and* $f_2 = \mathcal{L}^{-1}(F_2)$ *are piecewise continuous and of exponential order on* $[0, \infty)$. *Then*

$$\mathcal{L}^{-1}(c_1 F_1 + c_2 F_2) = c_1 \mathcal{L}^{-1}(F_1) + c_2 \mathcal{L}^{-1}(F_2)$$

for arbitrary constants c_1 *and* c_2.
In other words, the inverse Laplace transform is a linear operator.

TABLE 5.3.1 Elementary Laplace transforms.

	$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$	Notes
1.	1	$\frac{1}{s}, \quad s > 0$	Sec. 5.1; Ex. 4
2.	e^{at}	$\frac{1}{s-a}, \quad s > a$	Sec. 5.1; Ex. 5
3.	$t^n, \quad n = \text{positive integer}$	$\frac{n!}{s^{n+1}}, \quad s > 0$	Sec. 5.2; Cor. 5.2.5
4.	$t^p, \quad p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$	Sec. 5.1; Prob. 37
5.	$\sin at$	$\frac{a}{s^2+a^2}, \quad s > 0$	Sec. 5.1; Ex. 7
6.	$\cos at$	$\frac{s}{s^2+a^2}, \quad s > 0$	Sec. 5.1; Prob. 22
7.	$\sinh at$	$\frac{a}{s^2-a^2}, \quad s > \|a\|$	Sec. 5.1; Prob. 19
8.	$\cosh at$	$\frac{s}{s^2-a^2}, \quad s > \|a\|$	Sec. 5.1; Prob. 18
9.	$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}, \quad s > a$	Sec. 5.1; Prob. 23
10.	$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}, \quad s > a$	Sec. 5.1; Prob. 24
11.	$t^n e^{at}, \quad n = \text{positive integer}$	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$	Sec. 5.2; Prob. 8
12.	$u_c(t)$	$\frac{e^{-cs}}{s}, \quad s > 0$	Sec. 5.5; Eq. (4)
13.	$u_c(t)f(t-c)$	$e^{-cs}F(s)$	Sec. 5.5; Eq. (6)
14.	$e^{ct}f(t)$	$F(s-c)$	Sec. 5.2; Thm. 5.2.1
15.	$\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$	Sec. 5.6; Thm. 5.8.3
16.	$\delta(t-c)$	e^{-cs}	Sec. 5.7; Eq. (14)
17.	$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$	Sec. 5.2; Cor. 5.2.3
18.	$t^nf(t)$	$(-1)^nF^{(n)}(s)$	Sec. 5.2; Thm. 5.2.4

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Hence:

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{3}{s^2+2s+5}\right\}(t) &= 3\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+2^2}\right\}(t) = 3\mathcal{L}^{-1}\left\{\left.\frac{1}{s^2+2^2}\right|_{s\to s+1}\right\}(t) \\
&= 3e^{-t}\mathcal{L}^{-1}\left\{\frac{1}{s^2+2^2}\right\}(t) \\
&= \frac{3}{2}\,e^{-t}\mathcal{L}^{-1}\left\{\frac{2}{s^2+2^2}\right\}(t) \\
&= \frac{3}{2}\,e^{-t}\sin(2t)\,.
\end{aligned}$$

Partial fraction decomposition

Factor in denominator	Term in partial fraction decomposition
$ax+b$	$\dfrac{A}{ax+b}$
$(ax+b)^k$	$\dfrac{A_1}{ax+b}+\dfrac{A_2}{(ax+b)^2}+\cdots+\dfrac{A_k}{(ax+b)^k}$
ax^2+bx+c (without real roots)	$\dfrac{Ax+B}{ax^2+bx+c}$
$(ax^2+bx+c)^k$ (without real roots)	$\dfrac{A_1x+B_1}{ax^2+bx+c}+\dfrac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots\dfrac{A_kx+B_k}{(ax^2+bx+c)^k}$

Example:

$$\frac{x}{(x-1)(x-2)} = \frac{A}{x-1}+\frac{B}{x-2}$$

$$\frac{x^2+x+1}{(x-1)^2(x-2)} = \frac{A_1}{x-1}+\frac{A_2}{(x-1)^2}+\frac{B}{x-2}$$

$$\frac{x^2+x+1}{(x^2+1)(x-2)} = \frac{A_1x+B_1}{x^2+1}+\frac{A_2}{x-2}$$

$$\frac{x^2+x+1}{(x^2+1)^2(x-2)} = \frac{A_1x+B_1}{x^2+1}+\frac{A_2x+B_2}{(x^2+1)^2}+\frac{A_3}{x-2}$$