Section 5.4: Solving differential equation with Laplace transforms

Main Topics:

The Laplace transform method to solve initial value problems for
- second order linear DEs with constant coefficients
- higher order linear DEs with constant coefficients
- systems of first order linear DEs with constant coefficients
The Laplace transform method

From Sections 5.2 and 5.3: applying the Laplace transform to the IVP

\[ y'' + ay' + by = f(t) \]

with initial conditions \( y(0) = y_0, \ y'(0) = y_1 \)

leads to an algebraic equation for \( Y = \mathcal{L}\{y\} \), where \( y(t) \) is the solution of the IVP.

The algebraic equation can be solved for \( Y = \mathcal{L}\{y\} \).

Inverting the Laplace transform leads to the solution \( y = \mathcal{L}^{-1}\{Y\} \).

**Figure 5.0.1** Laplace transform method for solving differential equations.

From: J. Brannan & W. Joyce, Differential equations.
Example (continued from Section 5.2):

Solve the IVP: \( y'' - 3y' + 2y = e^{-3t} \) with initial conditions \( y(0) = 1, \ y'(0) = 0 \).
Example (continued from Section 5.2):

Solve the IVP: \( y'' - 3y' + 2y = e^{-3t} \) with initial conditions \( y(0) = 1, \ y'(0) = 0 \).

Recall our method:

Recall our method:

Recall our method:
Example (continued from Section 5.2):

Solve the IVP: \( y'' - 3y' + 2y = e^{-3t} \) with initial conditions \( y(0) = 1, \ y'(0) = 0 \).

Recall our method:

- Apply the Laplace transform to both sides of the DE:
  
  \[ \mathcal{L}\{y''\} - 3\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{e^{-3t}\} \]

  i.e.
  
  \[ [s^2\mathcal{L}\{y\}(s) - sy(0) - y'(0)] - 3[s\mathcal{L}\{y\}(s) - y(0)] + 2\mathcal{L}\{y\}(s) = \frac{1}{s + 3} \]

  i.e.
  
  \[ [s^2Y(s) - sy(0) - y'(0)] - 3[sY(s) - y(0)] + 2Y(s) = \frac{1}{s + 3} \]

  where \( Y(s) = \mathcal{L}\{y\}(s) \).
Example (continued from Section 5.2):

Solve the IVP: \( y'' - 3y' + 2y = e^{-3t} \) with initial conditions \( y(0) = 1, \ y'(0) = 0 \).

Recall our method:

- **Apply the Laplace transform to both sides of the DE:**
  \[
  \mathcal{L}\{y''\} - 3\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{e^{-3t}\}
  \]
  i.e.
  \[
  [s^2\mathcal{L}\{y\}(s) - sy(0) - y'(0)] - 3[s\mathcal{L}\{y\}(s) - y(0)] + 2\mathcal{L}\{y\}(s) = \frac{1}{s + 3}
  \]
  i.e.
  \[
  [s^2Y(s) - sy(0) - y'(0)] - 3[sY(s) - y(0)] + 2Y(s) = \frac{1}{s + 3}
  \]
  where \( Y(s) = \mathcal{L}\{y\}(s) \).

- **Insert the initial condition** \( y(0) = 1, \ y'(0) = 0 \):
  \[
  [s^2Y(s) - s] - 3[sY(s) - 1] + 2Y(s) = \frac{1}{s + 3}
  \]
  i.e.
  \[
  (s^2 - 3s + 2)Y(s) = s - 3 + \frac{1}{s + 3}
  \]
Example (continued from Section 5.2):

Solve the IVP: \( y'' - 3y' + 2y = e^{-3t} \) with initial conditions \( y(0) = 1 \), \( y'(0) = 0 \).

Recall our method:

- Apply the Laplace transform to both sides of the DE:

\[
\mathcal{L}\{y''\} - 3\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{e^{-3t}\}
\]

i.e.
\[
[s^2\mathcal{L}\{y\}(s) - sy(0) - y'(0)] - 3[s\mathcal{L}\{y\}(s) - y(0)] + 2\mathcal{L}\{y\}(s) = \frac{1}{s + 3}
\]

i.e.
\[
[s^2Y(s) - sy(0) - y'(0)] - 3[sY(s) - y(0)] + 2Y(s) = \frac{1}{s + 3}
\]

where \( Y(s) = \mathcal{L}\{y\}(s) \).

- Insert the initial condition \( y(0) = 1 \), \( y'(0) = 0 \):

\[
[s^2Y(s) - s] - 3[sY(s) - 1] + 2Y(s) = \frac{1}{s + 3}
\]

i.e.
\[
(s^2 - 3s + 2)Y(s) = s - 3 + \frac{1}{s + 3}
\]

- Solve for \( Y(s) \):

\[
Y(s) = \frac{s^2 - 8}{(s + 3)(s - 2)(s - 1)}
\]
Example (continued from Section 5.2):

Solve the IVP: \( y'' - 3y' + 2y = e^{-3t} \) with initial conditions \( y(0) = 1, \ y'(0) = 0 \).

Recall our method:

- Apply the Laplace transform to both sides of the DE:
  \[
  \mathcal{L}\{y''\} - 3 \mathcal{L}\{y'\} + 2 \mathcal{L}\{y\} = \mathcal{L}\{e^{-3t}\}
  \]
  i.e. \[
  [s^2 \mathcal{L}\{y\}(s) - sy(0) - y'(0)] - 3 [s \mathcal{L}\{y\}(s) - y(0)] + 2 \mathcal{L}\{y\}(s) = \frac{1}{s + 3}
  \]
  i.e. \[
  [s^2 Y(s) - sy(0) - y'(0)] - 3 [sY(s) - y(0)] + 2 Y(s) = \frac{1}{s + 3}
  \]
  where \( Y(s) = \mathcal{L}\{y\}(s) \).

- Insert the initial condition \( y(0) = 1, \ y'(0) = 0 \):
  \[
  [s^2 Y(s) - s] - 3 [sY(s) - 1] + 2 Y(s) = \frac{1}{s + 3}
  \]
  i.e. \[
  (s^2 - 3s + 2) Y(s) = s - 3 + \frac{1}{s + 3}
  \]

- Solve for \( Y(s) \):
  \[
  Y(s) = \frac{s^2 - 8}{(s + 3)(s - 2)(s - 1)}
  \]

Remark: \( s^2 - 3s + 2 \) is the characteristic polynomial of the DE \( y'' - 3y' + 2y = 0 \).
Compute $L^{-1}\{Y\}$ for $Y(s) = \frac{s^2 - 8}{(s+3)(s-2)(s-1)}$:

(1) *Partial fraction decomposition:*

$$\frac{s^2 - 8}{(s + 3)(s - 2)(s - 1)} = \frac{A}{s + 3} + \frac{B}{s - 2} + \frac{C}{s - 1}$$

is equivalent to

$$(A + B + C)s^2 - (-3A + 2B + C)s + (2A - 3B - 6C) = s^2 - 8.$$ Equating the coefficients of $s^2$, $s$ and 1 leads to a linear system of equations in $A, B, C$.

*Solution:* $A = \frac{1}{20}$, $B = -\frac{4}{5}$, $C = \frac{7}{4}$.

(2) *Linearity of $L^{-1}$:*

$$L^{-1}\{Y(s)\} = A L^{-1}\left\{\frac{1}{s + 3}\right\} + B L^{-1}\left\{\frac{1}{s - 2}\right\} + C L^{-1}\left\{\frac{1}{s - 1}\right\}.$$ 

(3) Look at the tables to find the inverse Laplace transforms: if $F(s) = \frac{1}{s-a}$ ($s > a$), then $L^{-1}\{F\}(t) = e^{at}$.

**Conclusion:** $y(t) = L^{-1}\{Y\} = \frac{1}{20} e^{-3t} - \frac{4}{5} e^{2t} + \frac{7}{4} e^t$. 
Constant coefficient linear DE’s of second order

In general, taking the Laplace transform of the initial value problem:

\[ ay'' + by' + cy = f \quad \text{with} \quad y(0) = y_0 \quad \text{and} \quad y'(0) = y_1 \]

gives

\[ a[s^2 Y(s) - sy(0) - y'(0)] + b[sY(s) - y(0)] + cY(s) = F(s) \]

where:

- \( Y(s) = \mathcal{L}\{y\}(s) \) is the Laplace transform of \( y \),
- \( F(s) = \mathcal{L}\{f\}(s) \) is the Laplace transform of \( f \).

It can be rewritten as

\[ (as^2 + bs + c)Y(s) - (as + b)y(0) - ay'(0) = F(s). \]

So,

\[ Y(s) = \frac{(as + b)y(0) + ay'(0)}{as^2 + bs + c} + \frac{F(s)}{as^2 + bs + c}. \]

The denominator \( as^2 + bs + c \) is the characteristic polynomial of \( ay'' + by' + cy = f \).

(Recall that \( as^2 + bs + c = 0 \) is its characteristic equation.)
Constant coefficient linear DE’s of arbitrary order

Generalize the above to IVP’s for constant coefficient linear DE’s of arbitrary order:

\[ a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(t) \]

with

\[ y^{(n-1)}(0) = y_{n-1}, \ y^{(n-2)}(0) = y_{n-2}, \ \cdots, \ y(0) = y_0. \]
Constant coefficient linear DE’s of arbitrary order

Generalize the above to IVP’s for constant coefficient linear DE’s of arbitrary order:

\[ a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(t) \]

with

\[ y^{(n-1)}(0) = y_{n-1}, \ y^{(n-2)}(0) = y_{n-2}, \ \cdots, \ y(0) = y_0. \]

- Apply the Laplace transform to both members of the DE and use that \( \mathcal{L} \) is a linear operator:

\[ a_n \mathcal{L}\{y^{(n)}\}(s) + a_{n-1} \mathcal{L}\{y^{(n-1)}\}(s) + \cdots + a_1 \mathcal{L}\{y'\}(s) + a_0 \mathcal{L}\{y\}(s) = \mathcal{L}\{f\}(s). \]
Constant coefficient linear DE’s of arbitrary order

Generalize the above to IVP’s for constant coefficient linear DE’s of arbitrary order:

\[ a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(t) \]

with

\[ y^{(n-1)}(0) = y_{n-1}, \quad y^{(n-2)}(0) = y_{n-2}, \quad \ldots, \quad y(0) = y_0. \]

- Apply the Laplace transform to both members of the DE and use that \( \mathcal{L} \) is a linear operator:

\[ a_n \mathcal{L}\{y^{(n)}\}(s) + a_{n-1} \mathcal{L}\{y^{(n-1)}\}(s) + \cdots + a_1 \mathcal{L}\{y'\}(s) + a_0 \mathcal{L}\{y\}(s) = \mathcal{L}\{f\}(s). \]

- RHS: compute \( \mathcal{L}\{f\}(s) \).

RHS: compute \( \mathcal{L}\{f\}(s) \).
Constant coefficient linear DE’s of arbitrary order

Generalize the above to IVP’s for constant coefficient linear DE’s of arbitrary order:

\[ a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(t) \]

with

\[ y^{(n-1)}(0) = y_{n-1}, \quad y^{(n-2)}(0) = y_{n-2}, \quad \ldots, \quad y(0) = y_0. \]

- Apply the Laplace transform to both members of the DE and use that \( \mathcal{L} \) is a linear operator:

\[ a_n \mathcal{L}\{y^{(n)}\}(s) + a_{n-1} \mathcal{L}\{y^{(n-1)}\}(s) + \cdots + a_1 \mathcal{L}\{y'\}(s) + a_0 \mathcal{L}\{y\}(s) = \mathcal{L}\{f\}(s). \]

- RHS: compute \( \mathcal{L}\{f\}(s) \).

- LHS: apply, for every \( k = 1, \ldots, n \):

\[
\mathcal{L}\{y^{(k)}\}(s) = s^k \underbrace{\mathcal{L}\{y\}(s)}_{Y(s)} - s^{k-1} y(0) - \cdots - sy^{(k-2)}(0) - y^{(k-1)}(0)
\]
Constant coefficient linear DE’s of arbitrary order

Generalize the above to IVP’s for constant coefficient linear DE’s of arbitrary order:

\[
a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(t)
\]

with

\[
y^{(n-1)}(0) = y_{n-1}, \quad y^{(n-2)}(0) = y_{n-2}, \quad \cdots, \quad y(0) = y_0.
\]

- Apply the Laplace transform to both members of the DE and use that \( \mathcal{L} \) is a linear operator:

\[
a_n \mathcal{L}\{y^{(n)}\}(s) + a_{n-1} \mathcal{L}\{y^{(n-1)}\}(s) + \cdots + a_1 \mathcal{L}\{y'\}(s) + a_0 \mathcal{L}\{y\}(s) = \mathcal{L}\{f\}(s).
\]

- RHS: compute \( \mathcal{L}\{f\}(s) \).
- LHS: apply, for every \( k = 1, \ldots, n \):

\[
\mathcal{L}\{y^{(k)}\}(s) = s^k \mathcal{L}\{y\}(s) - s^{k-1} y(0) - \cdots - sy^{(k-2)}(0) - y^{(k-1)}(0)
\]

\[
\underbrace{Y(s)}
\]

- The DE in \( y \) is transformed into an algebraic equation in \( Y \). Solve it for \( Y \).
Constant coefficient linear DE’s of arbitrary order

Generalize the above to IVP’s for constant coefficient linear DE’s of arbitrary order:

\[ a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(t) \]

with

\[ y^{(n-1)}(0) = y_{n-1}, \quad y^{(n-2)}(0) = y_{n-2}, \quad \ldots, \quad y(0) = y_0. \]

- Apply the Laplace transform to both members of the DE and use that \( \mathcal{L} \) is a linear operator:

\[
\begin{align*}
 a_n \mathcal{L}\{y^{(n)}\}(s) + a_{n-1} \mathcal{L}\{y^{(n-1)}\}(s) + \cdots + a_1 \mathcal{L}\{y'\}(s) + a_0 \mathcal{L}\{y\}(s) &= \mathcal{L}\{f\}(s).
\end{align*}
\]

- RHS: compute \( \mathcal{L}\{f\}(s) \).
- LHS: apply, for every \( k = 1, \ldots, n \):

\[
\mathcal{L}\{y^{(k)}\}(s) = s^k \mathcal{L}\{y\}(s) - s^{k-1} y(0) - \cdots - s y^{(k-2)}(0) - y^{(k-1)}(0)
\]

- The DE in \( y \) is transformed into an algebraic equation in \( Y \). Solve it for \( Y \).
- Compute \( y(t) = \mathcal{L}^{-1}\{Y(s)\} \): this is the solution of the initial IVP.
Example: Use the Laplace transform method to solve the IVP
\[ y''' + 3y' = \sin(2t) \quad \text{with} \quad y(0) = 1, \ y'(0) = 0, \ y''(0) = -2 \]
Example: Use the Laplace transform method to solve the IVP

\[ y'''' + 3y' = \sin(2t) \quad \text{with } y(0) = 1, \ y'(0) = 0, \ y''(0) = -2 \]

Apply the Laplace transform to both sides:

\[ \mathcal{L}\{y''''\}(s) + 3\mathcal{L}\{y'\}(s) = \mathcal{L}\{\sin(2t)\}(s) \]
**Example:** Use the Laplace transform method to solve the IVP

\[ y''' + 3y' = \sin(2t) \quad \text{with} \quad y(0) = 1, \ y'(0) = 0, \ y''(0) = -2 \]

Apply the Laplace transform to both sides:

\[ \mathcal{L}\{y''\}(s) + 3\mathcal{L}\{y\}'(s) = \mathcal{L}\{\sin(2t)\}(s) \]

Set \( Y(s) = \mathcal{L}\{y\}(s) \).
Example: Use the Laplace transform method to solve the IVP

\[ y''' + 3y' = \sin(2t) \quad \text{with } y(0) = 1, \ y'(0) = 0, \ y''(0) = -2 \]

Apply the Laplace transform to both sides:

\[ \mathcal{L}\{y'''}(s) + 3\mathcal{L}\{y'(s)\} = \mathcal{L}\{\sin(2t)\}(s) \]

Set \( Y(s) = \mathcal{L}\{y\}(s) \).

By the formula for the Laplace transform of the derivatives of \( y \):

\[
[s^3 Y(s) - s^2 y(0) - sy'(0) - y''(0)] + 3[sY(s) - y(0)] = \frac{2}{s^2 + 4}
\]
Example: Use the Laplace transform method to solve the IVP

\[ y''' + 3y' = \sin(2t) \quad \text{with } y(0) = 1, \ y'(0) = 0, \ y''(0) = -2 \]

Apply the Laplace transform to both sides:

\[ \mathcal{L}\{y'''}(s) + 3\mathcal{L}\{y'(s)\} = \mathcal{L}\{\sin(2t)(s)\} \]

Set \( Y(s) = \mathcal{L}\{y(s)\} \).

By the formula for the Laplace transform of the derivatives of \( y \):

\[ [s^3Y(s) - s^2y(0) - sy'(0) - y''(0)] + 3[sY(s) - y(0)] = \frac{2}{s^2 + 4} \]

Inserting the initial conditions:

\[ [s^3Y(s) - s^2 + 2] + 3sY(s) - 3 = \frac{2}{s^2 + 4} \]

i.e.
Example: Use the Laplace transform method to solve the IVP
\[ y''' + 3y' = \sin(2t) \quad \text{with } y(0) = 1, \ y'(0) = 0, \ y''(0) = -2 \]

Apply the Laplace transform to both sides:
\[ \mathcal{L}\{y'''}(s) + 3\mathcal{L}\{y'(s)\} = \mathcal{L}\{\sin(2t)\}(s) \]

Set \( Y(s) = \mathcal{L}\{y\}(s) \).

By the formula for the Laplace transform of the derivatives of \( y \):
\[
[s^3 Y(s) - s^2 y(0) - sy'(0) - y''(0)] + 3[sY(s) - y(0)] = \frac{2}{s^2 + 4}
\]

Inserting the initial conditions:
\[
[s^3 Y(s) - s^2 + 2] + 3sY(s) - 3 = \frac{2}{s^2 + 4}
\]

i.e.
\[
(s^3 + 3s)Y(s) = s^2 + 1 + \frac{2}{s^2 + 4}
\]
Example: Use the Laplace transform method to solve the IVP
\[ y''' + 3y' = \sin(2t) \quad \text{with} \quad y(0) = 1, \ y'(0) = 0, \ y''(0) = -2 \]

Apply the Laplace transform to both sides:
\[ \mathcal{L}\{y''\prime\prime\prime\}(s) + 3\mathcal{L}\{y'\}(s) = \mathcal{L}\{\sin(2t)\}(s) \]

Set \( Y(s) = \mathcal{L}\{y\}(s) \).

By the formula for the Laplace transform of the derivatives of \( y \):
\[ [s^3 Y(s) - s^2 y(0) - sy'(0) - y''(0)] + 3[sY(s) - y(0)] = \frac{2}{s^2 + 4} \]

Inserting the initial conditions:
\[ [s^3 Y(s) - s^2 + 2] + 3sY(s) - 3 = \frac{2}{s^2 + 4} \]
i.e.
\[ (s^3 + 3s)Y(s) = s^2 + 1 + \frac{2}{s^2 + 4} = \frac{s^4 + 5s^2 + 6}{s^2 + 4} \]
**Example:** Use the Laplace transform method to solve the IVP

\[ y''' + 3y' = \sin(2t) \quad \text{with } y(0) = 1, \ y'(0) = 0, \ y''(0) = -2 \]

Apply the Laplace transform to both sides:

\[ \mathcal{L}\{y'''}(s) + 3\mathcal{L}\{y'(s)\} = \mathcal{L}\{\sin(2t)\}(s) \]

Set \( Y(s) = \mathcal{L}\{y\}(s) \).

By the formula for the Laplace transform of the derivatives of \( y \):

\[ [s^3 Y(s) - s^2 y(0) - sy'(0) - y''(0)] + 3[sY(s) - y(0)] = \frac{2}{s^2 + 4} \]

Inserting the initial conditions:

\[ [s^3 Y(s) - s^2 + 2] + 3sY(s) - 3 = \frac{2}{s^2 + 4} \]

i.e.

\[ (s^3 + 3s)Y(s) = s^2 + 1 + \frac{2}{s^2 + 4} = \frac{s^4 + 5s^2 + 6}{s^2 + 4} = \frac{(s^2 + 3)(s^2 + 2)}{s^2 + 4} \]

Hence

\[ Y(s) = \frac{s^2 + 2}{s(s^2 + 4)} \]
Example: Use the Laplace transform method to solve the IVP
\[ y''' + 3y' = \sin(2t) \quad \text{with } y(0) = 1, \ y'(0) = 0, \ y''(0) = -2 \]

Apply the Laplace transform to both sides:
\[ \mathcal{L}\{y'''}(s) + 3\mathcal{L}\{y'(s) = \mathcal{L}\{\sin(2t)}(s) \]

Set \( Y(s) = \mathcal{L}\{y\}(s) \).

By the formula for the Laplace transform of the derivatives of \( y \):
\[ [s^3 Y(s) - s^2 y(0) - sy'(0) - y''(0)] + 3[sY(s) - y(0)] = \frac{2}{s^2 + 4} \]

Inserting the initial conditions:
\[ [s^3 Y(s) - s^2 + 2] + 3sY(s) - 3 = \frac{2}{s^2 + 4} \]
i.e.
\[ (s^3 + 3s)Y(s) = s^2 + 1 + \frac{2}{s^2 + 4} = \frac{s^4 + 5s^2 + 6}{s^2 + 4} = \frac{(s^2 + 3)(s^2 + 2)}{s^2 + 4} \]

Hence
\[ Y(s) = \frac{s^2 + 2}{s(s^2 + 4)} = \frac{1}{2} \frac{1}{s} + \frac{1}{2} \frac{s}{s^2 + 4} \]

by partial fraction decomposition.
Example: Use the Laplace transform method to solve the IVP

\[ y''' + 3y' = \sin(2t) \quad \text{with } y(0) = 1, \ y'(0) = 0, \ y''(0) = -2 \]

Apply the Laplace transform to both sides:

\[ \mathcal{L}\{y'''}(s) + 3\mathcal{L}\{y'(s) = \mathcal{L}\{\sin(2t)\}(s) \]

Set \( Y(s) = \mathcal{L}\{y\}(s) \).

By the formula for the Laplace transform of the derivatives of \( y \):

\[
[s^3 Y(s) - s^2 y(0) - sy'(0) - y''(0)] + 3[sY(s) - y(0)] = \frac{2}{s^2 + 4}
\]

Inserting the initial conditions:

\[
[s^3 Y(s) - s^2 + 2] + 3sY(s) - 3 = \frac{2}{s^2 + 4}
\]

i.e.

\[(s^3 + 3s)Y(s) = s^2 + 1 + \frac{2}{s^2 + 4} = \frac{s^4 + 5s^2 + 6}{s^2 + 4} = \frac{(s^2 + 3)(s^2 + 2)}{s^2 + 4}\]

Hence

\[
Y(s) = \frac{s^2 + 2}{s(s^2 + 4)} = \frac{1}{2} \frac{s^2 + 2}{s^2 + 4} = \frac{1}{2} \frac{s^2 + 2}{s^2 + 4} = \frac{1}{2} \frac{s^2 + 2}{s^2 + 4}
\]

by partial fraction decomposition. Thus:

\[
y(t) = \frac{1}{2} + \frac{1}{2} \cos(2t)
\]
Example: Use the Laplace transform method to solve the IVP

\[ y''' + 3y' = \sin(2t) \text{ with } y(0) = 1, \ y'(0) = 0, \ y''(0) = -2 \]

Apply the Laplace transform to both sides:

\[ \mathcal{L}\{y'''}(s) + 3\mathcal{L}\{y'(s) = \mathcal{L}\{\sin(2t)\}}(s) \]

Set \( Y(s) = \mathcal{L}\{y\}(s) \).

By the formula for the Laplace transform of the derivatives of \( y \):

\[ [s^3 Y(s) - s^2 y(0) - sy'(0) - y''(0)] + 3[sY(s) - y(0)] = \frac{2}{s^2 + 4} \]

Inserting the initial conditions:

\[ [s^3 Y(s) - s^2 + 2] + 3sY(s) - 3 = \frac{2}{s^2 + 4} \]

i.e.

\[ (s^3 + 3s)Y(s) = s^2 + 1 + \frac{2}{s^2 + 4} = \frac{s^4 + 5s^2 + 6}{s^2 + 4} = \frac{(s^2 + 3)(s^2 + 2)}{s^2 + 4} \]

Hence

\[ Y(s) = \frac{s^2 + 2}{s(s^2 + 4)} = \frac{1}{2} \frac{1}{s} + \frac{1}{2} \frac{s}{s^2 + 4} \]

by partial fraction decomposition. Thus:

\[ y(t) = \frac{1}{2} + \frac{1}{2} \cos(2t) = \cos^2 t \].
Laplace transform method for systems of 1st order linear DEs

Consider an IVP for a system of first-order constant coefficient linear DE's:

\[
\begin{align*}
  y'_1 &= a_{11}y_1 + a_{12}y_2 + f_1(t) \\
  y'_2 &= a_{21}y_1 + a_{22}y_2 + f_2(t)
\end{align*}
\]

with initial conditions

\[
\begin{align*}
  y_1(0) &= y_{10} \\
  y_2(0) &= y_{20}
\end{align*}
\]

Take the Laplace transform of each equation and set

\[
\begin{align*}
  Y_1 &= L\{y_1\} \\
  Y_2 &= L\{y_2\} \\
  F_1 &= L\{f_1\} \\
  F_2 &= L\{f_2\}
\end{align*}
\]

We get:

\[
\begin{align*}
  sY_1 - y_1(0) &= a_{11}Y_1 + a_{12}Y_2 + F_1(s) \\
  sY_2 - y_2(0) &= a_{21}Y_1 + a_{22}Y_2 + F_2(s)
\end{align*}
\]

This can be rewritten as

\[
\begin{align*}
  (s - a_{11})Y_1 - a_{12}Y_2 &= y_{10} + F_1(s) \\
  -a_{21}Y_1 + (s - a_{22})Y_2 &= y_{20} + F_2(s)
\end{align*}
\]

This is a system of two linear equations in \( Y_1 \) and \( Y_2 \).
Laplace transform method for systems of 1st order linear DEs

Consider an IVP for a system of first-order constant coefficient linear DE's :

\[
\begin{align*}
  y_1' &= a_{11}y_1 + a_{12}y_2 + f_1(t) \\
  y_2' &= a_{21}y_1 + a_{22}y_2 + f_2(t)
\end{align*}
\]

with initial conditions \( y_1(0) = y_{10}, y_2(0) = y_{20} \).
Laplace transform method for systems of 1st order linear DEs

Consider an IVP for a system of first-order constant coefficient linear DE's:

\[
\begin{align*}
    y'_1 &= a_{11} y_1 + a_{12} y_2 + f_1(t) \\
    y'_2 &= a_{21} y_1 + a_{22} y_2 + f_2(t)
\end{align*}
\]

with initial conditions \( y_1(0) = y_{10}, \ y_2(0) = y_{20} \).

Take the Laplace transform of each equation and set

\[
Y_1 = \mathcal{L}\{y_1\}, \quad Y_2 = \mathcal{L}\{y_2\}, \quad F_1 = \mathcal{L}\{f_1\}, \quad F_2 = \mathcal{L}\{f_2\}.
\]
Laplace transform method for systems of 1st order linear DEs

Consider an IVP for a system of first-order constant coefficient linear DE’s:

\[
\begin{align*}
    y_1' &= a_{11} y_1 + a_{12} y_2 + f_1(t) \\
    y_2' &= a_{21} y_1 + a_{22} y_2 + f_2(t)
\end{align*}
\]

with initial conditions \( y_1(0) = y_{10} \), \( y_2(0) = y_{20} \).

Take the Laplace transform of each equation and set

\[
Y_1 = \mathcal{L}\{y_1\}, \quad Y_2 = \mathcal{L}\{y_2\}, \quad F_1 = \mathcal{L}\{f_1\}, \quad F_2 = \mathcal{L}\{f_2\}.
\]

We get:

\[
\begin{align*}
    sY_1 - y_1(0) &= a_{11} Y_1 + a_{12} Y_2 + F_1(t) \\
    sY_2 - y_2(0) &= a_{21} Y_1 + a_{22} Y_2 + F_2(t).
\end{align*}
\]
Laplace transform method for systems of 1st order linear DEs

Consider an IVP for a system of first-order constant coefficient linear DE’s:

\[
\begin{align*}
    y'_1 &= a_{11} y_1 + a_{12} y_2 + f_1(t) \\
    y'_2 &= a_{21} y_1 + a_{22} y_2 + f_2(t)
\end{align*}
\]

with initial conditions \( y_1(0) = y_{10}, \ y_2(0) = y_{20} \).

Take the Laplace transform of each equation and set

\[
Y_1 = \mathcal{L}\{y_1\}, \quad Y_2 = \mathcal{L}\{y_2\}, \quad F_1 = \mathcal{L}\{f_1\}, \quad F_2 = \mathcal{L}\{f_2\}.
\]

We get:

\[
\begin{align*}
    sY_1 - y_1(0) &= a_{11} Y_1 + a_{12} Y_2 + F_1(t) \\
    sY_2 - y_2(0) &= a_{21} Y_1 + a_{22} Y_2 + F_2(t)
\end{align*}
\]

This can be rewritten as

\[
\begin{align*}
    (s - a_{11}) Y_1 - a_{12} Y_2 &= y_{10} + F_1(s) \\
    -a_{21} Y_1 + (s - a_{22}) Y_2 &= y_{20} + F_2(s)
\end{align*}
\]
Laplace transform method for systems of 1st order linear DEs

Consider an IVP for a system of first-order constant coefficient linear DE’s:

\[
\begin{align*}
    y'_1 &= a_{11}y_1 + a_{12}y_2 + f_1(t) \\
    y'_2 &= a_{21}y_1 + a_{22}y_2 + f_2(t)
\end{align*}
\]

with initial conditions \(y_1(0) = y_{10}, \ y_2(0) = y_{20}\).

Take the Laplace transform of each equation and set

\[
\begin{align*}
    Y_1 &= \mathcal{L}\{y_1\}, \quad Y_2 = \mathcal{L}\{y_2\}, \quad F_1 = \mathcal{L}\{f_1\}, \quad F_2 = \mathcal{L}\{f_2\}.
\end{align*}
\]

We get:

\[
\begin{align*}
    sY_1 - y_1(0) &= a_{11}Y_1 + a_{12}Y_2 + F_1(t) \\
    sY_2 - y_2(0) &= a_{21}Y_1 + a_{22}Y_2 + F_2(t).
\end{align*}
\]

This can be rewritten as

\[
\begin{align*}
    (s - a_{11})Y_1 - a_{12}Y_2 &= y_{10} + F_1(s) \\
    -a_{21}Y_1 + (s - a_{22})Y_2 &= y_{20} + F_2(s).
\end{align*}
\]

This is a system of two linear equations in \(Y_1\) and \(Y_2\).
Remark: The matrix form of
\[
\begin{align*}
(s - a_{11})Y_1 - a_{12}Y_2 &= y_{10} + F_1(s) \\
-a_{21}Y_1 + (s - a_{22})Y_2 &= y_{20} + F_2(s)
\end{align*}
\]
is
\[
(sI - A)Y = y_0 + F(s)
\]
where
\[A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad F(s) = \begin{pmatrix} F_1(s) \\ F_2(s) \end{pmatrix}, \quad y_0 = \begin{pmatrix} y_{10} \\ y_{20} \end{pmatrix}, \quad Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}\]
Remark: The matrix form of
\[
\begin{align*}
(s - a_{11})Y_1 - a_{12} Y_2 &= y_{10} + F_1(s) \\
-a_{21} Y_1 + (s - a_{22}) Y_2 &= y_{20} + F_2(s)
\end{align*}
\]
is
\[
(sI - A)Y = y_0 + F(s)
\]
where
\[
A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad F(s) = \begin{pmatrix} F_1(s) \\ F_2(s) \end{pmatrix}, \quad y_0 = \begin{pmatrix} y_{10} \\ y_{20} \end{pmatrix}, \quad Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}
\]

If the matrix \( sI - A \) is invertible, then we can solve for \( Y \):
\[
Y = (sI - A)^{-1} y_0 + (sI - A)^{-1} F(s)
\]
Remark: The matrix form of
\[
\begin{align*}
(s - a_{11})Y_1 - a_{12}Y_2 &= y_{10} + F_1(s) \\
-a_{21}Y_1 + (s - a_{22})Y_2 &= y_{20} + F_2(s)
\end{align*}
\]
is
\[
(sI - A)Y = y_0 + F(s)
\]
where
\[
A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad F(s) = \begin{pmatrix} F_1(s) \\ F_2(s) \end{pmatrix}, \quad y_0 = \begin{pmatrix} y_{10} \\ y_{20} \end{pmatrix}, \quad Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}
\]
If the matrix \( sI - A \) is invertible, then we can solve for \( Y \):
\[
Y = (sI - A)^{-1}y_0 + (sI - A)^{-1}F(s)
\]
Having determined \( Y_1 \) and \( Y_2 \), the solution of of the given IVP is:
\[
\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \left( \mathcal{L}^{-1}\{Y_1\} \right) \cdot \left( \mathcal{L}^{-1}\{Y_2\} \right).
\]
Remark: The matrix form of
\[
\begin{align*}
(s - a_{11})Y_1 - a_{12} Y_2 &= y_{10} + F_1(s) \\
-a_{21} Y_1 + (s - a_{22}) Y_2 &= y_{20} + F_2(s)
\end{align*}
\]
is
\[
(sI - A)Y = y_0 + F(s)
\]
where
\[
A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad F(s) = \begin{pmatrix} F_1(s) \\ F_2(s) \end{pmatrix}, \quad y_0 = \begin{pmatrix} y_{10} \\ y_{20} \end{pmatrix}, \quad Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}
\]

If the matrix $sI - A$ is invertible, then we can solve for $Y$:
\[
Y = (sI - A)^{-1} y_0 + (sI - A)^{-1} F(s)
\]

Having determined $Y_1$ and $Y_2$, the solution of the given IVP is:
\[
\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \left( \mathcal{L}^{-1}\{Y_1\} \right) \left( \mathcal{L}^{-1}\{Y_2\} \right)
\]

Example: Use the Laplace transform method to solve the initial value problem
\[
\begin{align*}
y_1' &= -y_1 + y_2 + e^t \\
y_2' &= y_1 - y_2 + e^t
\end{align*}
\]
with $y_1(0) = 1$ and $y_2(0) = 1$
Solve \[
\begin{cases}
    y_1' = -y_1 + y_2 + e^t \\
y_2' = y_1 - y_2 + e^t
\end{cases}
\] with \( y_1(0) = 1 \) and \( y_2(0) = 1 \).
Solve \[
\begin{align*}
\begin{cases}
y_1' &= -y_1 + y_2 + e^t \\
y_2' &= y_1 - y_2 + e^t
\end{cases}
\end{align*}
\]
with \( y_1(0) = 1 \) and \( y_2(0) = 1 \)

Apply the Laplace transform to both equations:

\[
\begin{align*}
\mathcal{L}\{y_1'\} &= -\mathcal{L}\{y_1\} + \mathcal{L}\{y_2\} + \mathcal{L}\{e^t\} \\
\mathcal{L}\{y_2'\} &= \mathcal{L}\{y_1\} - \mathcal{L}\{y_2\} + \mathcal{L}\{e^t\}
\end{align*}
\]
Solve \[
\begin{align*}
    y'_1 &= -y_1 + y_2 + e^t \\
    y'_2 &= y_1 - y_2 + e^t
\end{align*}
\] with \(y_1(0) = 1\) and \(y_2(0) = 1\)

Apply the Laplace transform to both equations:
\[
\begin{align*}
    \mathcal{L}\{y'_1\} &= -\mathcal{L}\{y_1\} + \mathcal{L}\{y_2\} + \mathcal{L}\{e^t\} \\
    \mathcal{L}\{y'_2\} &= \mathcal{L}\{y_1\} - \mathcal{L}\{y_2\} + \mathcal{L}\{e^t\}
\end{align*}
\]

Set \(Y_1 = \mathcal{L}\{y_1\}\) and \(Y_2 = \mathcal{L}\{y_2\}\). Then

\[
\begin{align*}
    sY_1(s) - y_1(0) &= -Y_1(s) + Y_2(s) + 1 \\
    sY_2(s) - y_2(0) &= Y_1(s) - Y_2(s) + 1
\end{align*}
\]

Hence

\[
\begin{align*}
    sY_1(s) - 1 &= -Y_1(s) + Y_2(s) + 1 \\
    sY_2(s) - 1 &= Y_1(s) - Y_2(s) + 1
\end{align*}
\]

\[
\begin{align*}
    (s+1)Y_1(s) &= 1 + 1 - Y_1(s) + Y_2(s) + 1 \\
    (s+1)Y_2(s) &= Y_1(s) - Y_2(s) + 1 + 1
\end{align*}
\]

This system of linear equations has solutions \(Y_1(s) = Y_2(s) = \frac{1}{s-1}\) (for \(s > 1\)).

Taking their inverse Laplace transform:

\[
\begin{align*}
    y_1(t) &= y_2(t) = \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} = e^t.
\end{align*}
\]
Solve \[
\begin{align*}
y_1' &= -y_1 + y_2 + e^t \\
y_2' &= y_1 - y_2 + e^t
\end{align*}
\] with \(y_1(0) = 1\) and \(y_2(0) = 1\)

Apply the Laplace transform to both equations:

\[
\begin{align*}
\mathcal{L}\{y_1'\} &= -\mathcal{L}\{y_1\} + \mathcal{L}\{y_2\} + \mathcal{L}\{e^t\} \\
\mathcal{L}\{y_2'\} &= \mathcal{L}\{y_1\} - \mathcal{L}\{y_2\} + \mathcal{L}\{e^t\}
\end{align*}
\]

Set \(Y_1 = \mathcal{L}\{y_1\}\) and \(Y_2 = \mathcal{L}\{y_2\}\). Then

\[
\begin{align*}
sY_1(s) - y_1(0) &= -Y_1(s) + Y_2(s) + \frac{1}{s-1} \\
sY_2(s) - y_2(0) &= Y_1(s) - Y_2(s) + \frac{1}{s-1}
\end{align*}
\] (for \(s > 1\)).
Solve \[
\begin{cases}
  y'_1 = -y_1 + y_2 + e^t \\
  y'_2 = y_1 - y_2 + e^t
\end{cases}
\] with \( y_1(0) = 1 \) and \( y_2(0) = 1 \)

Apply the Laplace transform to both equations:
\[
\begin{align*}
\mathcal{L}\{y'_1\} &= -\mathcal{L}\{y_1\} + \mathcal{L}\{y_2\} + \mathcal{L}\{e^t\} \\
\mathcal{L}\{y'_2\} &= \mathcal{L}\{y_1\} - \mathcal{L}\{y_2\} + \mathcal{L}\{e^t\}
\end{align*}
\]

Set \( Y_1 = \mathcal{L}\{y_1\} \) and \( Y_2 = \mathcal{L}\{y_2\} \). Then
\[
\begin{align*}
  sY_1(s) - y_1(0) &= -Y_1(s) + Y_2(s) + \frac{1}{s-1} \\
  sY_2(s) - y_2(0) &= Y_1(s) - Y_2(s) + \frac{1}{s-1}
\end{align*}
\]
(\text{for } s > 1).

Hence
\[
\begin{align*}
  sY_1(s) - 1 &= -Y_1(s) + Y_2(s) + \frac{1}{s-1} \\
  sY_2(s) - 1 &= Y_1(s) - Y_2(s) + \frac{1}{s-1}
\end{align*}
\]

Taking their inverse Laplace transform:
\[
\begin{align*}
y_1(t) &= y_2(t) = \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} = e^t
\end{align*}
\]
Solve \[
\begin{align*}
y'_1 &= -y_1 + y_2 + e^t \\
y'_2 &= y_1 - y_2 + e^t
\end{align*}
\] with \(y_1(0) = 1\) and \(y_2(0) = 1\)

Apply the Laplace transform to both equations:

\[
\begin{align*}
\mathcal{L}\{y'_1\} &= -\mathcal{L}\{y_1\} + \mathcal{L}\{y_2\} + \mathcal{L}\{e^t\} \\
\mathcal{L}\{y'_2\} &= \mathcal{L}\{y_1\} - \mathcal{L}\{y_2\} + \mathcal{L}\{e^t\}
\end{align*}
\]

Set \(Y_1 = \mathcal{L}\{y_1\}\) and \(Y_2 = \mathcal{L}\{y_2\}\). Then

\[
\begin{align*}
sY_1(s) - y_1(0) &= -Y_1(s) + Y_2(s) + \frac{1}{s-1} \\
sY_2(s) - y_2(0) &= Y_1(s) - Y_2(s) + \frac{1}{s-1}
\end{align*}
\] (for \(s > 1\)).

Hence

\[
\begin{align*}
sY_1(s) - 1 &= -Y_1(s) + Y_2(s) + \frac{1}{s-1} \\
sY_2(s) - 1 &= Y_1(s) - Y_2(s) + \frac{1}{s-1}
\end{align*}
\]
i.e.

\[
\begin{align*}
(s + 1)Y_1(s) - Y_2(s) &= 1 + \frac{1}{s-1} \\
-Y_1(s) + (s + 1)Y_2(s) &= 1 + \frac{1}{s-1}
\end{align*}
\]
Solve \[ \begin{cases} y_1' = -y_1 + y_2 + e^t \\ y_2' = y_1 - y_2 + e^t \end{cases} \] with \( y_1(0) = 1 \) and \( y_2(0) = 1 \)

Apply the Laplace transform to both equations:

\[ \begin{cases} \mathcal{L}\{y_1'\} = -\mathcal{L}\{y_1\} + \mathcal{L}\{y_2\} + \mathcal{L}\{e^t\} \\ \mathcal{L}\{y_2'\} = \mathcal{L}\{y_1\} - \mathcal{L}\{y_2\} + \mathcal{L}\{e^t\} \end{cases} \]

Set \( Y_1 = \mathcal{L}\{y_1\} \) and \( Y_2 = \mathcal{L}\{y_2\} \). Then

\[ \begin{cases} sY_1(s) - y_1(0) = -Y_1(s) + Y_2(s) + \frac{1}{s-1} \\ sY_2(s) - y_2(0) = Y_1(s) - Y_2(s) + \frac{1}{s-1} \end{cases} \quad \text{for } s > 1. \]

Hence

\[ \begin{cases} sY_1(s) - 1 = -Y_1(s) + Y_2(s) + \frac{1}{s-1} \\ sY_2(s) - 1 = Y_1(s) - Y_2(s) + \frac{1}{s-1} \end{cases}, \]

i.e.

\[ \begin{cases} (s + 1)Y_1(s) - Y_2(s) = 1 + \frac{1}{s-1} \\ -Y_1(s) + (s + 1)Y_2(s) = 1 + \frac{1}{s-1} \end{cases}. \]

This system of linear equations has solutions

\[ Y_1(s) = Y_2(s) = \frac{1}{s-1} \quad \text{for } s > 1. \]
Solve \[
\begin{align*}
    y_1' &= -y_1 + y_2 + e^t \\
    y_2' &= y_1 - y_2 + e^t
\end{align*}
\] with \(y_1(0) = 1\) and \(y_2(0) = 1\).

Apply the Laplace transform to both equations:
\[
\begin{align*}
    \mathcal{L}\{y_1'\} &= -\mathcal{L}\{y_1\} + \mathcal{L}\{y_2\} + \mathcal{L}\{e^t\} \\
    \mathcal{L}\{y_2'\} &= \mathcal{L}\{y_1\} - \mathcal{L}\{y_2\} + \mathcal{L}\{e^t\}
\end{align*}
\]

Set \(Y_1 = \mathcal{L}\{y_1\}\) and \(Y_2 = \mathcal{L}\{y_2\}\). Then
\[
\begin{align*}
    sY_1(s) - y_1(0) &= -Y_1(s) + Y_2(s) + \frac{1}{s-1} \\
    sY_2(s) - y_2(0) &= Y_1(s) - Y_2(s) + \frac{1}{s-1}
\end{align*}
\]
(for \(s > 1\)).

Hence
\[
\begin{align*}
    sY_1(s) - 1 &= -Y_1(s) + Y_2(s) + \frac{1}{s-1} \\
    sY_2(s) - 1 &= Y_1(s) - Y_2(s) + \frac{1}{s-1}
\end{align*}
\]

i.e.
\[
\begin{align*}
    (s + 1)Y_1(s) - Y_2(s) &= 1 + \frac{1}{s-1} \\
    -Y_1(s) + (s + 1)Y_2(s) &= 1 + \frac{1}{s-1}
\end{align*}
\]

This system of linear equations has solutions
\[
Y_1(s) = Y_2(s) = \frac{1}{s-1} \quad \text{(for } s > 1\text{)}.
\]

Taking their inverse Laplace transform:
\[
y_1(t) = y_2(t) = \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} = e^t
\]