Section 5.5: Discontinuous functions and periodic functions

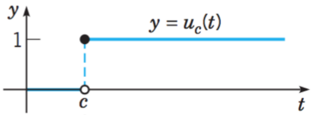
Main Topics:

- **Unit step functions,**
- **indicator functions,**
- **translates of functions,**
- **periodic functions,**
- **and their Laplace transforms.**

Unit step functions

Definition

For a real number c, the **unit step function** u_c is defined by $u_c(t) = \begin{cases} 0 & \text{if } t < c \\ 1 & \text{if } t \geq c \end{cases}$.



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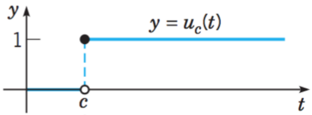


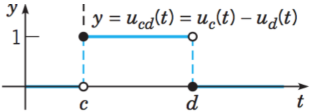
- $u_c(t)$ is piecewise continuous but not continuous.
- When $c = 0$, the unit step function is known as the **Heaviside function**.
- The definition of the value of u_c at the jump discontinuity $t = c$ is immaterial. We could just not define it at all at c.

Indicator functions

Definition

For real numbers $c < d$, the **indicator function for the interval** $[c, d)$ is the function u_{cd} defined by

$$u_{cd}(t) = \begin{cases} 0 & \text{if } t < c \text{ or } t \geq d \\ 1 & \text{if } c \leq t < d \end{cases}$$



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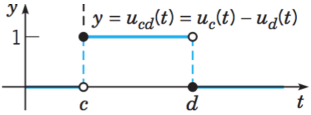

- u_{cd} is piecewise continuous but is not continuous.
- As for u_c, the value of u_{cd} at the jump discontinuities is immaterial. We could just not define them at all at c and d.

Representation of piecewise continuous functions

Example:

(1) Use the unit step functions to give a representation of the piecewise continuous function

$$f(t) = \begin{cases} t & \text{if } 0 \leq t < 2 \\ 1 & \text{if } 2 \leq t < 3 \\ e^{-2t} & \text{if } t \geq 3 \end{cases}$$

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Since $u_0(t) = 1$ for all $t \ge 0$, when we restrict ourselves to $t \ge 0$, we can write:

$$f(t) = t - (t-1)u_2(t) + (e^{-2t} - 1)u_3(t) \qquad (t \ge 0)\,.$$

Laplace transforms of u_c and u_{cd}

- For $s > 0$

$$\mathcal{L}\{u_c\}(s) = \frac{e^{-cs}}{s}$$

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Remark: For $s > 0$ we have $\mathcal{L}\{u_0\}(s) = \frac{1}{s} = \mathcal{L}\{1\}(s)$.
This is correct because $u_0(t) = 1$ for $t > 0$.

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- Consequence: for $s > 0$

$$\mathcal{L}\{u_{cd}\}(s) = \mathcal{L}\{u_c\}(s) - \mathcal{L}\{u_d\}(s) = \frac{e^{-cs} - e^{-ds}}{s}.$$

Translate of a function

Definition

Fix $c \geq 0$ a real number and let f be a function defined for $t \geq 0$. The **translate of** f is the function g defined by

$$g(t) = \begin{cases} 0 & \text{if } t < c \\ f(t-c) & \text{if } t \geq c \end{cases} = u_c(t)f(t-c)$$

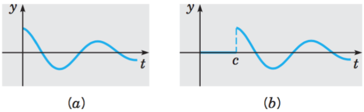

FIGURE 5.5.3 A translation of the given function. (a) $y = f(t)$; (b) $y = u_c(t)\, f(t-c)$.

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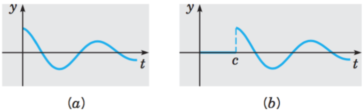

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Theorem (Theorem 5.5.1)

Suppose $\mathcal{L}\{f\}(s)$ exists for $s > a \geq 0$. Let $c \geq 0$. Then for $s > a \geq 0$,

$$\mathcal{L}\{u_c(t) f(t-c)\}(s) = e^{-cs} \mathcal{L}\{f\}(s)\,.$$

Example:

(1) Compute the Laplace transform of $f(t) = \begin{cases} t & \text{if } 0 < t < 2 \\ 1 & \text{if } 2 \leq t < 3 \\ e^{-2t} & \text{if } t \geq 3 \end{cases}$.

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We have found (see p. 4):

$$f(t) = tu_0(t) - (t-1)u_2(t) + (e^{-2t} - 1)u_3(t) \,.$$

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$$f_0(t) = t, \qquad f_2(t) = t+1, \qquad f_3(t) = e^{-2(t+3)} - 1 = e^{-6}e^{-2t} - 1.$$

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&= \mathcal{L}\{f_0(t)\}(s) + e^{-2s}\mathcal{L}\{f_2(t)\}(s) + e^{-3s}\mathcal{L}\{f_3(t)\}(s) \\
&= \frac{1}{s^2} - e^{-2s}\frac{1+s}{s^2} + e^{-3s}\Big(\frac{e^{-6}}{s+2} - \frac{1}{s}\Big)
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Hence

$$f(t) = \begin{cases} t & \text{if } 0 \leq t < 2 \\ 2 & \text{if } t \geq 2 \end{cases}$$

Periodic functions

Definition (Definition 5.5.2)

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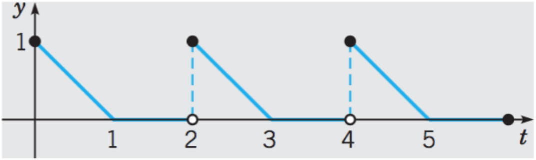
- The sine and cosine functions are 2π-periodic, while the tangent function is π-periodic.
- $f(t) = \begin{cases} 1-t & \text{if } 0 \le t < 1 \\ 0 & \text{if } 1 \le t < 2 \end{cases}$ can be turned into a 2-periodic function as follows:



A key property of T-periodic functions is that they can be studied only on any interval of length T. For this, it is convenient to introduce the **window function** $f_T(t)$ associated with f:

$$f_T(t) = f(t)(1 - u_T(t)) = \begin{cases} f(t) & \text{if } 0 \leq t \leq T \\ 0 & \text{otherwise} \end{cases}$$

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Compute the Laplace transform of the 2-periodic function f defined on $[0, 2)$ by

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Here $T = 2$ and $F_2(s) = \int_0^2 f_T(t)\,dt = \int_0^1 te^{-st}\,dt = \frac{1 - e^{-s}}{s^2} - \frac{e^{-s}}{s}$.

So, $F(s) = \mathcal{L}\{f\}(s) = \dfrac{1 - e^{-s}}{s^2(1 - e^{-2s})} - \dfrac{e^{-s}}{s(1 - e^{-2s})}$.

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Thus:

$$f(t) = \mathcal{L}^{-1}\{F\}(t) = \begin{cases} 1 & \text{if } 0 \leq t < 1 \\ 0 & \text{if } 1 \leq t < 2 \end{cases} \quad \text{extended to } \mathbb{R} \text{ by 2-periodicity}$$