Section 5.6: Differential equations with discontinuous forcing functions

Main Topics:

Examples of differential equations with constant coefficients

$$ay'' + by' + cy = f(t)$$

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in which the nonhomogenous term f (=the forcing function in a spring-mass system) is **not continuous**.

General fact: even if f is not continuous but **piecewise continuous**, then the solution y and also y' are still continuous (while y'' has, as f, jump discontinuities).

This fact suitably extends to constant coefficient DE's of order > 2 as well.

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Solve the initial value problem: $y''(t) = u_c(t)$ with initial condition $y(0) = 0$, $y'(0) = 0$, where $c > 0$.

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Integrate both sides of the DE on $[0, t]$:

$$y'(t) - y'(0) = \int_0^t y''(\tau)\, d\tau = \int_0^t u_c(\tau)\, d\tau = \begin{cases} 0 & \text{if } 0 \le t < c \\ \int_c^t d\tau = t - c & \text{if } t \ge c \end{cases},$$

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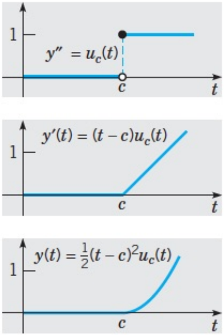
Integrate once more on $[0, t]$:

$$y(t) - y(0) = \int_0^t y'(\tau)\, d\tau = \int_0^t (\tau - c)u_c(\tau)\, d\tau$$

$$= \begin{cases} 0 & \text{if } 0 \le t < c \\ \int_c^t (\tau - c)\, d\tau = \left[\frac{1}{2}\tau^2 - c\tau\right]_{\tau=c}^{\tau=t} = \frac{1}{2}(t-c)^2 & \text{if } t \ge c \end{cases}.$$

Since $y(0) = 0$, we conclude $y(t) = \frac{1}{2}(t - c)^2 u_c(t)$.



The graphs of y, y', y'' show the smoothing effect of integration:

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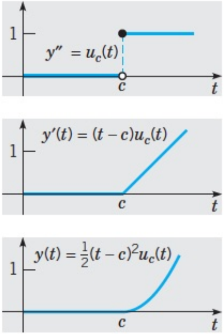

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Find the solution of the IVP:

$$2y'' + y' + 2y = u_{12} = u_1 - u_2, \qquad y(0) = y'(0) = 0\,.$$

This could be a model of the motion of a dumped oscillator subject an external force u_{12}.

Apply the Laplace transform method: $2\mathcal{L}\{y''\} + \mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{u_1\} - \mathcal{L}\{u_2\}$,
i.e., with $Y = \mathcal{L}\{y\}$,

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$$(2s^2 + s + 2)Y(s) = \frac{e^{-s} - e^{-2s}}{s}, \quad \text{i.e.} \quad Y(s) = \frac{e^{-s} - e^{-2s}}{s(2s^2 + s + 2)}\,.$$

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It remains to find h.

$$H(s) = \frac{1}{s(2s^2 + s + 2)}$$

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Thus, by the table of Laplace transforms,

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h(t) = \mathcal{L}^{-1}\{H(s)\} = \frac{1}{2} - \frac{1}{2}\left[e^{-t/4}\cos\left(\frac{\sqrt{15}}{4}t\right) + e^{-t/4}\frac{\sqrt{15}}{15}\sin\left(\frac{\sqrt{15}}{4}t\right)\right]
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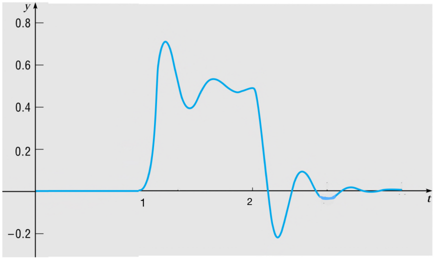
Plot of the solution:



For $0 \leq t \leq 1$: we have $u_{12}(t) = 0$ and the IVP becomes

$$2y'' + y' + 2y = 0, \quad y(0) = y'(0) = 0\,.$$

The unique solution is $y = 0$.

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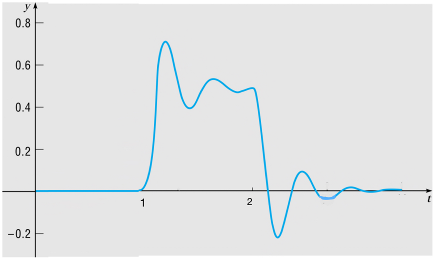

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For $1 \le t \le 2$: we have $u_{12}(t) = 1$ and the IVP becomes

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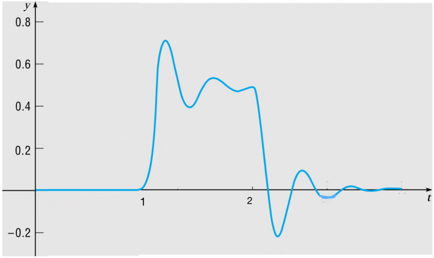

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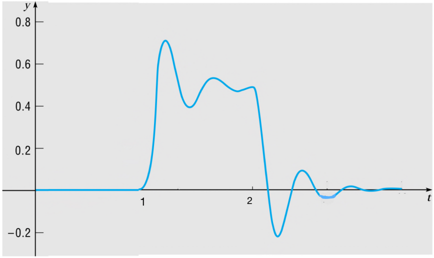

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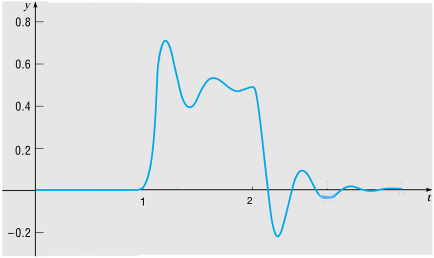
$h(t)$ is the solution of the IVP: $2y'' + y' + 2y = 1, \qquad y(0) = y'(0) = 0\,.$
Shifting the initial conditions at $t = 1$ corresponds to shifting the solution to $y(t) = u_1(t)h(t-1)$ on $[1, 2]$.

Conclusion: $y(t) = u_1(t)h(t-1) - u_2(t)h(t-2)$ where

$$h(t) = \frac{1}{2} - \frac{1}{2}e^{-t/4}\left[\cos\left(\frac{\sqrt{15}}{4}t\right) + \frac{\sqrt{15}}{15}\sin\left(\frac{\sqrt{15}}{4}t\right)\right]$$

Plot of the solution:



For $0 \le t \le 1$: we have $u_{12}(t) = 0$ and the IVP becomes

$$2y''+y'+2y = 0, \quad y(0) = y'(0) = 0\,.$$

The unique solution is $y = 0$.

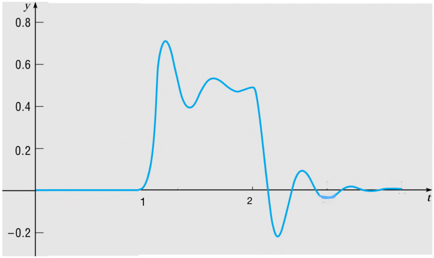
For $1 \le t \le 2$: we have $u_{12}(t) = 1$ and the IVP becomes

$$2y'' + y' + 2y = 1, \qquad y(1) = y'(1) = 0\,.$$

$h(t)$ is the solution of the IVP: $\;2y'' + y' + 2y = 1, \qquad y(0) = y'(0) = 0\,.$
Shifting the initial conditions at $t = 1$ corresponds to shifting the solution to $y(t) = u_1(t)h(t-1)$ on $[1, 2]$.

The external force 1 is positive, so the motion will start in the positive direction at $t = 1$. Then it oscillates (with damping) around 1/2.

Plot of the solution:

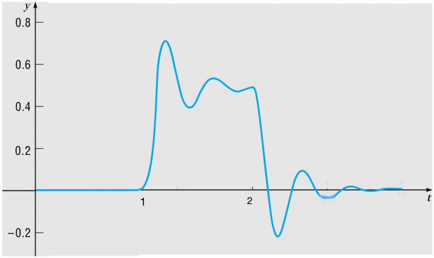


For $t \leq 2$: we have $u_{12}(t) = 0$ and the IVP becomes

$$2y'' + y' + 2y = 0, \qquad y(2) = *,\ y'(2) = **,$$

where $*$ and $**$ can be computed from the solution on $[1, 2]$,
e.g. $y(2) = \lim_{t\to 2} u_1(t)h(t-1)$.

Plot of the solution:



For $t \leq 2$: we have $u_{12}(t) = 0$ and the IVP becomes

$$2y'' + y' + 2y = 0, \qquad y(2) = *, \ y'(2) = **,$$

where $*$ and $**$ can be computed from the solution on $[1, 2]$,
e.g. $y(2) = \lim_{t \to 2} u_1(t)h(t-1)$.

There is dumping and no external force : the motion keeps oscillating and tend to dye out for $t \to +\infty$.