# Section 6.1: Systems of first order linear equations

**Main topics:**

- Systems of $n$ linear first order DE's,
- Matrix notation.
- Applications to $n$-th order linear DE.

A system of $n$ linear first-order DEs is of the form:

$$\begin{cases} x_1' = p_{11}(t)x_1 + p_{12}(t)x_2 + \cdots + p_{1n}(t)x_n + g_1(t) \\ x_2' = p_{21}(t)x_1 + p_{22}(t)x_2 + \cdots + p_{2n}(t)x_n + g_2(t) \\ \qquad \vdots \\ x_n' = p_{n1}(t)x_1 + p_{n2}(t)x_2 + \cdots + p_{nn}(t)x_n + g_n(t) \end{cases}$$

Initial conditions: $x_1(t_0) = x_{01}, \ldots, x_n(t_0) = x_{0n}$.

**Matrix notation:**

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t)$$

Initial condition: $\mathbf{x}(t_0) = \mathbf{x}_0$

where

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \quad \mathbf{P}(t) = \begin{pmatrix} p_{11}(t) & \cdots & p_{1n}(t) \\ \vdots & \ddots & \vdots \\ p_{n1}(t) & \cdots & p_{nn}(t) \end{pmatrix}, \quad \mathbf{g}(t) = \begin{pmatrix} g_1(t) \\ \vdots \\ g_n(t) \end{pmatrix}, \quad \mathbf{x}_0 = \begin{pmatrix} x_{01} \\ \vdots \\ x_{0n} \end{pmatrix}$$

**Example:**

Verify that $\mathbf{x}(t) = e^{-t}\begin{pmatrix} 6 \\ -8 \\ -4 \end{pmatrix} + 2e^{2t}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ satisfies the system of linear differential equations

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \mathbf{x}$$

# Linear *n*-th order equations

A linear *n*-th order DE: $y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)' + p_n(t)y = g(t)$
can be transformed into a system of $n$ first-order linear DE.

*Change of variables:* $x_1 = y,\ x_2 = y',\ x_3 = y'',\ \ldots, x_n = y^{(n-1)}$.

We obtain

$$\begin{cases} x_1' = y' = x_2 \\ x_2' = y'' = x_3 \\ \qquad \vdots \\ x_{n-1}' = y^{(n-1)} = x_n \\ x_n' = -p_n(t)x_1 - p_{n-1}(t)x_2 - \cdots - p_1(t)x_n + g(t)\,. \end{cases}$$

**Matrix notation:**

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -p_n(t) & -p_{n-1}(t) & -p_{n-2}(t) & \cdots & -p_1(t) \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ g(t) \end{pmatrix}$$

**Remark:** the size of the matrix is the order of the DE.

The change of variables $x_1 = y,\ x_2 = y',\ x_3 = y'',\ \ldots, x_n = y^{(n-1)}$ transforms the IVP:

$$y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)' + p_n(t)y = g(t)$$

with initial conditions:

$$y(t_0) = y_0, y'(t_0) = y_1, \ldots, y^{(n-1)}(t_0) = y_{n-1}$$

into the IVP:

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -p_n(t) & -p_{n-1}(t) & -p_{n-2}(t) & \cdots & -p_1(t) \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ g(t) \end{pmatrix} \quad \text{with} \quad \mathbf{x}(t_0) = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{pmatrix}.$$

**Example:** Transform the differential equation

$$y^{(4)} + 6y''' + 3y = t$$

into an equivalent first order linear system.