Section 6.2: Basic theory of systems of n first order linear equations

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t)$$

where

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \quad \mathbf{P}(t) = \begin{pmatrix} p_{11}(t) & \cdots & p_{1n}(t) \\ \vdots & \ddots & \vdots \\ p_{n1}(t) & \cdots & p_{nn}(t) \end{pmatrix}, \quad \mathbf{g}(t) = \begin{pmatrix} g_1(t) \\ \vdots \\ g_n(t) \end{pmatrix},$$

Main topics:

- Existence and unicity of solutions
- Principle of superposition (homogenous systems, i.e. $\mathbf{g}(t) = \mathbf{0}$)
- Independence of solutions and the Wronskian (homogeneous systems)
- General solutions (homogeneous systems)

Two generalizations with respect to Section 3.3:

- n linear DE
- $\mathbf{P}(t)$ not necessarily constant in t.

Existence and unicity of solutions

Consider the system of n linear first-order DEs: $\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t)$ where

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Remarks:

- These are **matrix-valued functions**, or simply **matrix functions**, i.e. matrices or vectors whose entries are functions of t.
- A matrix function is **continuous** on the interval I if all its entries are continuous functions of t.

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Example:

$\mathbf{P}(t) = \begin{pmatrix} 1/(t-1) & e^t \\ t^2 & \ln t \end{pmatrix}$ is a continuous matrix function on $I = (0, 1)$ or on $I = (1, +\infty)$.

Theorem (Theorem 6.2.1)

*If $\mathbf{P}(t)$ and $\mathbf{g}(t)$ are continuous matrix functions on a open interval I and $t_0 \in I$, then the IVP: $\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t)$ with initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$ has a **unique solution** $\mathbf{x} = \mathbf{x}(t)$ with t in I.*

The principle of superposition (for homogenous linear systems)

Definition

We say that $\mathbf{x}$ is a **linear combination** of $\mathbf{x_1}, \ldots, \mathbf{x_k}$, written

$$\mathbf{x} = c_1\mathbf{x_1} + \cdots + c_k\mathbf{x_k},$$

if there are constants $c_1, \ldots, c_k$ such that $\mathbf{x}(t) = c_1\mathbf{x_1}(t) + \cdots + c_k\mathbf{x_k}(t)$ for all t.

Theorem (Theorem 6.2.2, Principle of superposition)

*If $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_k$ are k solutions of the **homogeneous** linear system*

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$$

on an open interval I, then any linear combination $c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_k\mathbf{x}_k$, where the c_j's are constants, is a solution of the system in I.

The Wronskian

Let I be an open interval. For $t \in I$ consider

$$\mathbf{x_1}(t) = \begin{pmatrix} x_{11}(t) \\ \vdots \\ x_{n1}(t) \end{pmatrix}, \ldots, \mathbf{x_n}(t) = \begin{pmatrix} x_{1n}(t) \\ \vdots \\ x_{nn}(t) \end{pmatrix}.$$

Let

$$\mathbf{X}(t) = [\mathbf{x_1}(t), \ldots, \mathbf{x_n}(t)] = \begin{pmatrix} x_{11}(t) & x_{12}(t) & \cdots & x_{1n}(t) \\ x_{21}(t) & x_{22}(t) & \cdots & x_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1}(t) & x_{n2}(t) & \cdots & x_{nn}(t) \end{pmatrix}$$

be the $n \times n$ matrix-valued function on I having $\mathbf{x_1}(t), \ldots, \mathbf{x_n}(t)$ as column vectors.

Definition

The **Wronskian** of $\mathbf{x_1}, \ldots, \mathbf{x_n}$ is the function

$$W[\mathbf{x_1}, \ldots \mathbf{x_n}] : t \mapsto W[\mathbf{x_1}, \ldots \mathbf{x_n}](t) = \det(\mathbf{X}(t))$$

defined for $t \in I$.

Definition

The vector-valued functions $\mathbf{x_1} = \begin{pmatrix} x_{11} \\ \vdots \\ x_{n1} \end{pmatrix}, \ldots, \mathbf{x_n} = \begin{pmatrix} x_{1n} \\ \vdots \\ x_{nn} \end{pmatrix}$ defined on the interval I are said **linearly dependent** on I if there are constants $c_1, \ldots, c_n$ (not all zero and *independent of* $t \in I$) such that

$$c_1 \mathbf{x_1}(t) + \cdots + c_n \mathbf{x_n}(t) = 0 \text{ for all } t \in I.$$

If $\mathbf{x_1}, \ldots, \mathbf{x_n}$ are not linearly dependent, we say they are **linearly independent**. This means that $c_1 = \cdots = c_n = 0$ are the only constants such that

$$c_1 \mathbf{x_1}(t) + \cdots + c_n \mathbf{x_n}(t) = 0 \text{ for all } t \in I.$$

Example: $\mathbf{x_1} = \begin{pmatrix} 1+t \\ t \\ 1-t \end{pmatrix}, \mathbf{x_2} = \begin{pmatrix} 3 \\ t+2 \\ t \end{pmatrix}, \mathbf{x_3} = \begin{pmatrix} 1-2t \\ 2-t \\ 3t-2 \end{pmatrix}$ are linearly dependent on the interval $I = (-\infty, \infty)$. Indeed

$$2\begin{pmatrix} 1+t \\ t \\ 1-t \end{pmatrix} - \begin{pmatrix} 3 \\ t+2 \\ t \end{pmatrix} + \begin{pmatrix} 1-2t \\ 2-t \\ 3t-2 \end{pmatrix} = \begin{pmatrix} 2+2t-3+1-2t \\ 2t-t-2+2-t \\ 2-2t-t+3t-2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

So the above relation holds for all $t \in (-\infty, \infty)$ when $c_1 = 2$, $c_2 = -1$ and $c_3 = 1$.

Fundamental sets of solutions of $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$

Let $\mathbf{x_1}, \ldots, \mathbf{x_n}$ be n solutions of the **homogeneous** linear system $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ of n linear DE on the open interval I.

Theorem (Theorem 6.2.5)

Suppose that $\mathbf{P}(t)$ is a continuous function of $t \in I$:

- *If $\mathbf{x_1}, \ldots, \mathbf{x_n}$ are linearly independent on I, then $W[\mathbf{x_1}, \ldots \mathbf{x_n}](t) \neq 0$ for all $t \in I$.*
- *If $\mathbf{x_1}, \ldots, \mathbf{x_n}$ are linearly dependent on I, then $W[\mathbf{x_1}, \ldots \mathbf{x_n}](t) = 0$ for all $t \in I$.*

Consequence: if there is $t_0 \in I$ such that $W[\mathbf{x_1}, \ldots \mathbf{x_n}](t_0) \neq 0$, then $\mathbf{x_1}, \ldots, \mathbf{x_n}$ are linearly independent on I.

Definition

Let $\mathbf{x_1}, \ldots, \mathbf{x_n}$ be a set of n solutions of the homogeneous linear system $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ of n linear DEs.
If $\mathbf{x_1}, \ldots, \mathbf{x_n}$ are linearly independent on I, then we say that they form a **fundamental set of solutions** on I.

General solution of $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$

Theorem (Theorem 6.2.6)

Let $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ *be a homogeneous system of n linear equations, where* $\mathbf{P}(t)$ *is continuous on the interval I.*
Let $\mathbf{x_1}, \ldots, \mathbf{x_n}$ *be a fundamental set of solutions of* $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ *on I.*

Then any solution of the this system on I is of the form

$$\mathbf{x} = c_1\mathbf{x_1} + \cdots + c_n\mathbf{x_n}$$

for some constants $c_1, \ldots, c_n$. *This is the* ***general solution*** *of the system.*

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Theorem (Theorem 6.2.6)

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Then any solution of the this system on I is of the form

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*for some constants $c_1, \ldots, c_n$. This is the **general solution** of the system.*

Moreover: an intitial condition $\mathbf{x}(t_0) = \mathbf{x}_0$, where $\mathbf{x}_0 = \begin{pmatrix} x_{10} \\ \vdots \\ x_{n0} \end{pmatrix}$ is a constant vector, uniquely determines the constants $c_1, \cdots, c_n$. The solution to the IVP is unique.

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Conclusion:

- The general solution of a **homogenous** system of n linear first order DEs (with $\mathbf{P}(t)$ continuous) is a linear combination of n **linearly independent** solutions **(same** n**)**
- To find the general solution it is enough to find n linearly independent solutions.
- To check if n solutions $\mathbf{x_1}, \ldots, \mathbf{x_n}$ are linearly independent, we compute their Wronskian.

Example:

Consider the system $\mathbf{x}' = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 0 & 2 \\ -1 & -2 & 4 \end{pmatrix} \mathbf{x}$ on the interval $I = \mathbb{R}$.

One can verify that the functions $\mathbf{x}_1(t) = e^t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{x}_2(t) = e^{2t} \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ and

$\mathbf{x}_3(t) = e^{3t} \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$ are solutions of this system.

- Computing their Wronskian, determine whether $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$ form a fundamental set of solutions for this system of DE on $\mathbb{R}$.
- Write down the general solution.
- Determine the unique solution satisfying the initial condition $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$