# Section 6.3: Homogenous linear systems with constant coeffs.

Consider the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ where $A$ is a $n \times n$ matrix with real coefficients.

## Theorem (Theorem 6.3.1 – cf. Section 3.3 for $n = 2$)

*Suppose that:*

(1) $\mathbf{A}$ *has* ***real*** *(not necessarily distinct) eigenvalues* $\lambda_1, \cdots, \lambda_n$*,*

(2) $\mathbf{A}$ *has eigenvectors* $\mathbf{v}_1, \cdots, \mathbf{v}_n$ *associated with the eigenvalues* $\lambda_1, \cdots, \lambda_n$*, respectively, so that* $\mathbf{v}_1, \cdots, \mathbf{v}_n$ *are* ***linearly independent****.*

*Then the vector functions*

$$\mathbf{x}_1(t) = e^{\lambda_1 t}\mathbf{v}_1, \ldots, \mathbf{x}_n(t) = e^{\lambda_n t}\mathbf{v}_n$$

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*The* ***general solution*** *of* $\mathbf{x}' = \mathbf{A}\mathbf{x}$ *on* $\mathbb{R}$ *is*

$$\mathbf{x}(t) = C_1 e^{\lambda_1 t}\mathbf{v}_1 + \cdots + C_n e^{\lambda_n t}\mathbf{v}_n$$

*where* $t \in \mathbb{R}$ *and* $C_1, \ldots, C_n$ *are constants.*

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*where* $t \in \mathbb{R}$ *and* $C_1, \ldots, C_n$ *are constants.*

- (2) is satisfied if the eigenvalues $\lambda_1, \ldots, \lambda_n$ are all distinct (Corollary 6.3.2).
- both (1) and (2) are satisfied if $\mathbf{A}$ is **symmetric**, i.e. $\mathbf{A} = \mathbf{A}^T$, where $\mathbf{A}^T$ denotes the transpose of $\mathbf{A}$.

**Example:**

Find the general solution of the system of linear differential equations $\mathbf{x}' = \mathbf{A}\mathbf{x}$ where

$$A = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}$$

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*General solution:* $\mathbf{x}(t) = C_1 e^{8t} \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} + C_2 e^{-t} \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + C_3 e^{-t} \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$ where $C_1, C_2, C_3 \in \mathbb{R}$.

## Definition

We say that an $n \times n$ real matrix $\mathbf{A}$ is **nondefective** if there is a set of $n$ linearly independent vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ which are eigenvectors of $\mathbf{A}$.
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- If $\mathbf{A}$ has $n$ distinct eigenvalues, then the corresponding eigenvectors are $n$ linearly independent vectors. So $\mathbf{A}$ is nondefective.
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In the next section we will look at the general solution of the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ where $\mathbf{A}$ is nondefective but its eigenvalues are not necessarily real.