Section 6.4: Nondefective matrices with complex eigenvalues

Consider the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ where $\mathbf{A}$ is a $n \times n$ with real coefficients and nondefective.

Suppose that not all eigenvalues of $\mathbf{A}$ are real.

Section 6.4: Nondefective matrices with complex eigenvalues

Consider the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ where $\mathbf{A}$ is a $n \times n$ with real coefficients and nondefective.

Suppose that not all eigenvalues of $\mathbf{A}$ are real.

Since $\mathbf{A}$ has real coefficients, as in the case of 2×2-matrices, we have:

- for any complex eigenvalue $\lambda = \mu + i\nu$ of $\mathbf{A}$, also $\overline{\lambda}$ is an eigenvalue of $\mathbf{A}$.
- if $\mathbf{v} = \mathbf{a} + i\mathbf{b}$ is eigenvector of $\mathbf{A}$ of eigenvalue λ, then
  - $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$ is a complex-valued solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$.
  - the real-valued vector functions

$$\mathrm{Re}\,\mathbf{x}(t) \quad \text{and} \quad \mathrm{Im}\,\mathbf{x}(t)$$

    are linearly independent solutions of $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

Theorem

Let $\mathbf{A}$ *be an* $n \times n$ *nondefective matrix. Suppose that* $\mathbf{A}$ *has:*

- *complex (not real) eigenvalues* $\lambda_1, \overline{\lambda_1}, \ldots, \lambda_m, \overline{\lambda_m}$
- *real eigenvalues* $\lambda_{2m+1}, \ldots, \lambda_n$

Let

$$\mathbf{v}_1, \overline{\mathbf{v}_1}, \ldots, \mathbf{v}_m, \overline{\mathbf{v}_m} \qquad \text{and} \qquad \mathbf{v}_{2m+1}, \ldots, \mathbf{v}_n$$

be the n corresponding linearly independent eigenvectors (they exist as $\mathbf{A}$ *is nondefective).*

Then the general solution of the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ *is*

$$\mathbf{x}(t) = C_1 \operatorname{Re} \mathbf{x}_1(t) + C_2 \operatorname{Im} \mathbf{x}_1(t) + \cdots + C_{2m-1} \operatorname{Re} \mathbf{x}_m(t) + C_{2m} \operatorname{Im} \mathbf{x}_m(t) + C_{2m+1}\mathbf{x}_{2m+1}(t) + \cdots + C_n \mathbf{x}_n(t)\,,$$

where

- *for* $j = 1, \ldots, m$: $\mathbf{x}_j = e^{t\lambda_j}\mathbf{v}_j$ *is the complex-valued solutions corresponding to the complex eigenvalue* λ_j *and its complex eigenvector* $\mathbf{v}_j$
- *for* $j = 2m+1, \ldots, n$: $\mathbf{x}_j = e^{t\lambda_j}\mathbf{v}_j$ *is the solution corresponding to the real eigenvalue* λ_j *and its real eigenvector* $\mathbf{v}_j$.

Example: Find the general solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ where $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$

Example: Find the general solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ where $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$

Characteristic equation: $\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & -\lambda & 1 \\ 0 & -1 & -\lambda \end{vmatrix} = 0$ i.e. $(1-\lambda)(\lambda^2+1) = 0$

Eigenvalues $\lambda_1 = i$, $\lambda_2 = -i$, $\lambda_3 = 1$.

Eigenvectors

for $\lambda_1 = i$: $\mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix}$ and for $\lambda_3 = 1$: $\mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

Example: Find the general solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ where $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$

Characteristic equation: $\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & -\lambda & 1 \\ 0 & -1 & -\lambda \end{vmatrix} = 0$ i.e. $(1-\lambda)(\lambda^2+1) = 0$

Eigenvalues $\lambda_1 = i$, $\lambda_2 = -i$, $\lambda_3 = 1$.
Eigenvectors

for $\lambda_1 = i$: $\mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix}$ and for $\lambda_3 = 1$: $\mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

Complex-valued solution associated with $\lambda_1 = i$:

$$\mathbf{x}_1(t) = e^{it}\mathbf{v}_1 = e^{it}\begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix} = \begin{pmatrix} 0 \\ e^{it} \\ ie^{it} \end{pmatrix} = \begin{pmatrix} 0 \\ \cos t + i\sin t \\ -\sin t + i\cos t \end{pmatrix} = \begin{pmatrix} 0 \\ \cos t \\ -\sin t \end{pmatrix} + i\begin{pmatrix} 0 \\ \sin t \\ \cos t \end{pmatrix}$$

with

$$\operatorname{Re}\mathbf{x}_1(t) = \begin{pmatrix} 0 \\ \cos t \\ -\sin t \end{pmatrix} \quad \text{and} \quad \operatorname{Im}\mathbf{x}_1(t) = \begin{pmatrix} 0 \\ \sin t \\ \cos t \end{pmatrix}.$$

Solution corresponding to the real eigenvalue $\lambda_3 = 1$: $\mathbf{x}_3(t) = e^t\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

Example: Find the general solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ where $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$

Characteristic equation: $\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & -\lambda & 1 \\ 0 & -1 & -\lambda \end{vmatrix} = 0$ i.e. $(1-\lambda)(\lambda^2+1) = 0$

Eigenvalues $\lambda_1 = i$, $\lambda_2 = -i$, $\lambda_3 = 1$.
Eigenvectors

for $\lambda_1 = i$: $\mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix}$ and for $\lambda_3 = 1$: $\mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

Complex-valued solution associated with $\lambda_1 = i$:

$$\mathbf{x}_1(t) = e^{it}\mathbf{v}_1 = e^{it}\begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix} = \begin{pmatrix} 0 \\ e^{it} \\ ie^{it} \end{pmatrix} = \begin{pmatrix} 0 \\ \cos t + i\sin t \\ -\sin t + i\cos t \end{pmatrix} = \begin{pmatrix} 0 \\ \cos t \\ -\sin t \end{pmatrix} + i\begin{pmatrix} 0 \\ \sin t \\ \cos t \end{pmatrix}$$

with

$$\operatorname{Re}\mathbf{x}_1(t) = \begin{pmatrix} 0 \\ \cos t \\ -\sin t \end{pmatrix} \quad \text{and} \quad \operatorname{Im}\mathbf{x}_1(t) = \begin{pmatrix} 0 \\ \sin t \\ \cos t \end{pmatrix}.$$

Solution corresponding to the real eigenvalue $\lambda_3 = 1$: $\mathbf{x}_3(t) = e^t\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

General solution:

$$\mathbf{x}(t) = C_1\operatorname{Re}\mathbf{x}_1(t) + C_2\operatorname{Im}\mathbf{x}_1(t) + C_3\mathbf{x}_3(t) = C_1\begin{pmatrix} 0 \\ \cos t \\ \sin t \end{pmatrix} + C_2\begin{pmatrix} 0 \\ \sin t \\ \cos t \end{pmatrix} + C_3e^t\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

where C_1, C_2, C_3 are constant, $t \in \mathbb{R}$.