# Chapter 7: Nonlinear DE and Stability

## Section 7.1: Autonomous Systems and Stability

**Main Topics:**

- **Autonomous systems**
- **Stability and asymptotic stability:**
  precise mathematical definitions of stable, asymptotically stable, and unstable equilibrium solutions.
- **Basins of attraction**
- **The oscillating pendulum.**

Recall from section 3.6:

- A system of DE $\begin{cases} \dfrac{dx}{dt} = F(x,y) \\ \dfrac{dy}{dt} = G(x,y) \end{cases}$ in which the functions $F$ and $G$ do not depend on the independent variable $t$ is said to be **autonomous**.
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- **Theorem 3.6.1:** Suppose both $F$ and $G$ are **continuous** functions of $(x, y)$ in a domain $D$ of the $xy$-plane and let $(x_0, y_0) \in D$. Then there is a unique solution of the system satisfying the initial condition $x(t_0) = x_0$ and $y(t_0) = y_0$. This solution is in general only defined for some values of $t$ in a small interval $I$ containing $t_0$.

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**Matrix notation:**

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}) \quad \text{with initial condition} \quad \mathbf{x}(t_0) = \mathbf{x}_0$$

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Setting

$$\mathbf{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

we can also write

$$\mathbf{x}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} \quad \text{and} \quad \mathbf{f}(\mathbf{x}) = F(x,y)\mathbf{i} + G(x,y)\mathbf{j}$$

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If $y = 0$, the first equation gives $2x = 0$, i.e. $x = 0$. This gives a critical point $(0, 0)$.

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If $y = 0$, the first equation gives $2x = 0$, i.e. $x = 0$. This gives a critical point $(0, 0)$.

If $x = 1$, the first equation gives $(1+y)(2+y) = 0$, i.e. $y = -1$ or $y = -2$.
This gives two critical points: $(1, -1)$, $(1, -2)$.

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The magnitude (or length) of $\mathbf{x}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$ is $\|\mathbf{x}(t)\| = \sqrt{x(t)^2 + y(t)^2}$.

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## Definition

A critical point $\mathbf{x}_0$ of the system $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ is said to be **stable** provided:
for any $\epsilon > 0$, there exists a $\delta > 0$ such that

- every solution $\mathbf{x} = \Phi(t)$ of the system which satisfies
$$\|\Phi(0) - \mathbf{x}_0\| < \delta$$
exists for all $t \geq 0$ **and**
- it satisfies
$$\|\Phi(t) - \mathbf{x}_0\| < \epsilon$$
for all $t \geq 0$.

A critical point which is not stable is said to be **unstable**.

**Remark:** roughly speaking, the second condition means that no matter how small we choose $\epsilon > 0$ we can find a (smaller) $\delta > 0$ so that all solutions that **start** “sufficiently close” (=within the distance $\delta$) to $\mathbf{x}_0$, **remain** within the distance $\epsilon$ from $\mathbf{x}_0$ for all times.

## Definition

A critical point $\mathbf{x}_0$ of the system $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ is said to be **asymptotically stable** if

- it is stable, and
- there exists $\delta > 0$ such that every solution $\mathbf{x} = \Phi(t)$ of the system which satisfies

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then $\lim_{t \to +\infty} \Phi(t) = \mathbf{x_0}$

**Remark:** all solutions that start “sufficiently close” (=within a distance $\delta$) to $\mathbf{x}_0$ must stay “close” to $\mathbf{x}_0$ **and** approach $\mathbf{x}_0$ as $t \to +\infty$.

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## Definition

Let $\mathbf{x}_0$ be an asymptotically stable critical point.

The **basin of attraction** of $\mathbf{x}_0$ is the set of all points $P$ in the $xy$-plane that have the property that a trajectory (solution) starting at $P$ approches $\mathbf{x}_0$ as $t \to +\infty$.

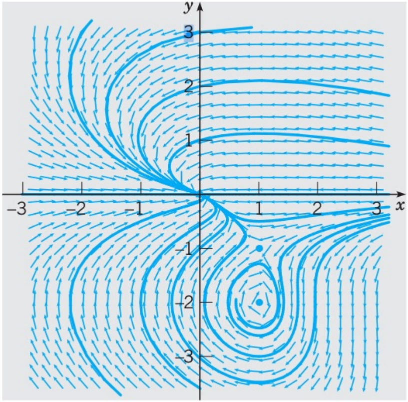
A trajectory that bounds a basin of attraction is called a **separatrix**.

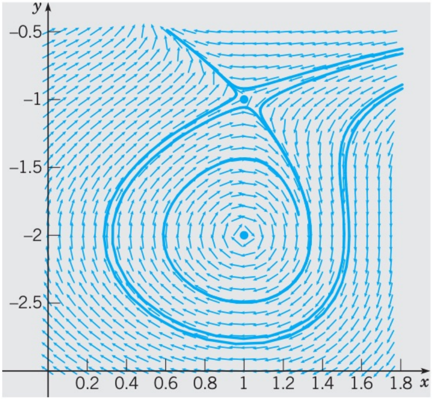
**Example:** Consider the system

$$dx/dt = -(x+y)(2+y) \qquad dy/dt = -y(1-x)$$

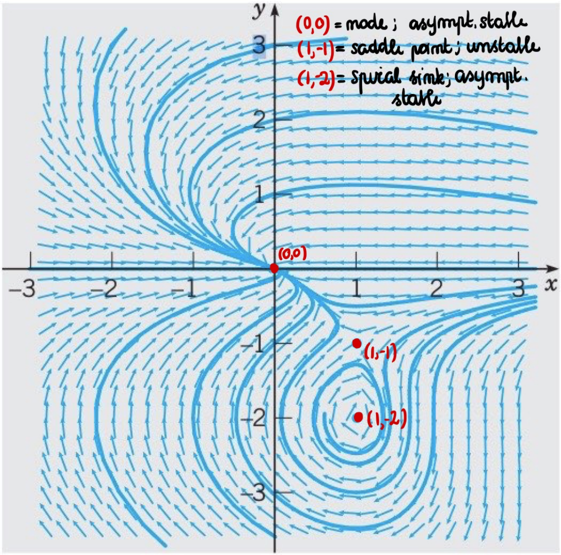
The critical points are $(0,0)$, $(1,-1)$ and $(1,-2)$.

- Using the drawing of the direction field and the phase portrait below to determine whether each critical point is stable, asymptotically stable or unstable.
- Determine the basins of attraction of the asymptotically stable critical points.

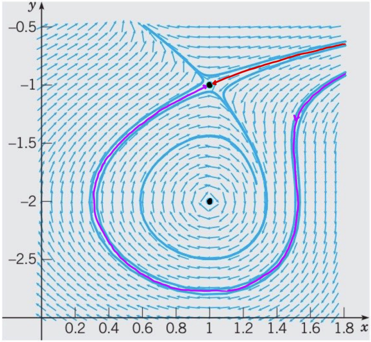


Type and stability of the critical points:

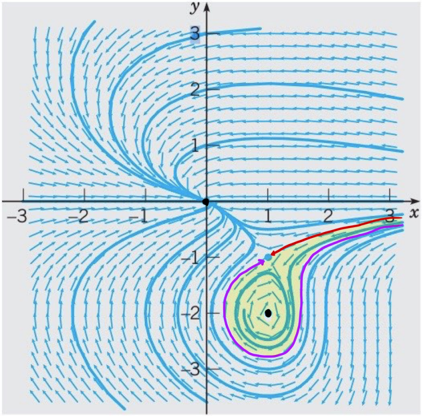


Separatrices




Basins of attractions of the asymtoptically stable critical points:

- the yellow area for $(1, -1)$;
- all the rest but the separatrices for $(0, 0)$



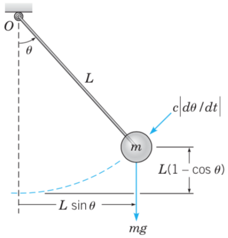
# The oscillating pendulum

The equation of motion of the oscillating pendulum represented below is

$$\frac{d^2\theta}{dt^2} + \gamma \frac{d\theta}{dt} + \omega^2 \sin\theta = 0$$

where

- $m$ is the mass attached to one end of a rigid but weighless rod,
- $L$ is the length of the rod,
- $\theta$, the unknown describing the motion, is the angle beween the rod and the vertical downward direction, with counterclockwise direction taken as positive.
- $\gamma = c/mL$ is the damping factor (a constant),
- $\omega^2 = g/L$ (with $mg$=weight of the mass $m$)



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Setting $x = \theta$ and $y = \frac{d\theta}{dt}$ transforms the pendulum equation into the autonomous system of 1st order DE:

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So: the critical points are the points $(x, y) = (\pm n\pi, 0)$, where $n$ is an integer.

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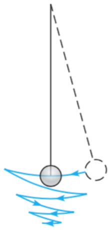
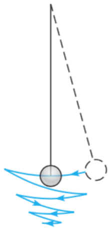
They corresponds to two positions along the vertical ($y = 0$): one below the point of support ($\theta = 0, 2\pi, ...$), the other above the point of support ($\theta = \pi, 3\pi, ...$).

Our intuition suggests that the first is stable and the second is unstable.



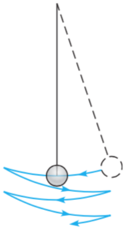
**With damping** (by air resistance for instance):



**With damping** (by air resistance for instance):

The critical point $(0, 0)$ (i.e. $\theta = 0$) is asymptotically stable : the pendulum will oscillate back and forth with decreasing amplitude as the energy is dissipated by the damping force. The mass will eventually reach the equilibrium position.

The same applies to the critical points $(n\pi, 0)$ with $n$ even, which are also asymptotically stable.



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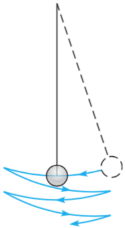
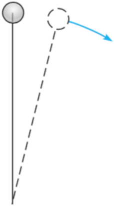


**Without damping**:

The critical point $(0,0)$ (that is $\theta = 0$) is stable but not asymptotically stable: there is no dissipation and the pendulum will oscillate indefinitely with a constant amplitude. The mass remains close to the equilibrium but will never reach it.

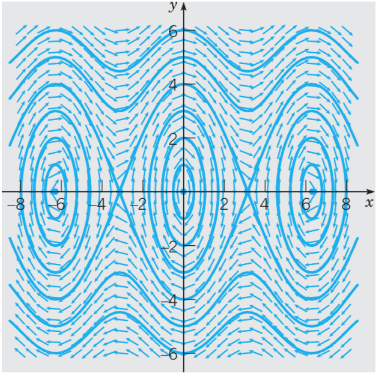
The same applies to the critical points $(n\pi, 0)$ with $n$ even, which are also stable but not asymptotically stable if there is no damping.



The critical point $(\pi, 0)$ (i.e. $\theta = \pi$) is unstable, whether the pendulum is damped or not (by air resistance): the slightest perturbation will cause the mass to fall under the effect of gravity. The pendulum will ultimately approach the lower equilibrium position $(\theta = 0)$.

The same applies to the critical points $(n\pi, 0)$ with $n$ odd, which are also unstable (with or without damping).

## **Example:** **Undamped Pendulum with $\omega = 2$:**




Stable critical points $(n\pi, 0)$ with $n$ even, but no asymptotically stable critical points.

Unstable critical points $(n\pi, 0)$ with $n$ odd.

The curves connecting different saddle points are called **separatrices** because they separates regions of periodic motions along closed ellipses and regions where the motion oscillates along “cos"-like curves.