Section 7.3: Competing species

Main Topics:

- **Logistic equations,**
- **Mathematical models of competitive situtations:**
  two populations competing for scarse resources.

  Examples: two species competing for food supplies, competitions in a same economic market
- **Long time behavior of solutions near the critical solutions**

  Method: linear approximations.

A logistic model for competing species

Let $x(t)$ and $y(t)$ denote the size at time t of two interacting or competing populations for a common food or resource.

We will consider the following modelization (based on the logistic equation):

$$\begin{aligned} \frac{dx}{dt} &= x(\epsilon_1 - \sigma_1 x - \alpha_1 y) \\ \frac{dy}{dt} &= y(\epsilon_2 - \sigma_2 y - \alpha_2 x) \end{aligned}$$

where

- ϵ_1 and ϵ_2 are the **growth rates** of the two populations,
- ϵ_1/σ_1 and ϵ_2/σ_2 are their **saturation levels**,
- α_1 measures the **interference** of the species y with the species x,
- α_2 measures the **interference** of the species x with the species y.

Example:

- We want to study the qualitative behavior of the solutions of the competing system

$$\frac{dx}{dt} = x(1 - x - y)$$
$$\frac{dy}{dt} = y(0.75 - y - 0.5x)$$

- The critical points of the system are:

$$(0,0),\ (0,0.75),\ (1,0) \text{ and } (0.5,0.5)$$

Critical points and direction field for the system

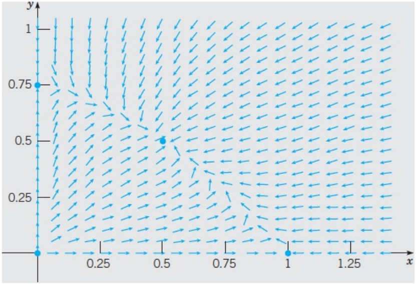

FIGURE 7.3.1 Critical points and direction field for the system (3).

For the above system, the functions

$$F(x, y) = x(1 - x - y) \text{ and } G(x, y) = y(0.75 - y - 0.5x)$$

have continuous partial derivatives of any order (in particular, up to order 2).

The system is almost linear in the neighborhood of any of the critical points.

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The Jacobian matrix of the system is

$$\mathbf{J} = \begin{pmatrix} 1 - 2x - y & -x \\ -0.5y & 0.75 - 2y - 0.5x \end{pmatrix}$$

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The linear approximation of the system at the critical point (X, Y) is

$$\frac{d}{dt}\begin{pmatrix} u \\ w \end{pmatrix} = \mathbf{J}(X, Y)\begin{pmatrix} u \\ w \end{pmatrix}$$

where $u = x - X$ and $w = y - Y$.

$$\mathbf{J}(\mathbf{x}, \mathbf{y}) = \begin{pmatrix} 1 - 2x - y & -x \\ -0.5y & 0.75 - 2y - 0.5x \end{pmatrix}$$

$$\mathbf{J(x, y)} = \begin{pmatrix} 1 - 2x - y & -x \\ -0.5y & 0.75 - 2y - 0.5x \end{pmatrix}$$

Case of $(0, 0)$**:**
This solution corresponds to the extinction of both species.

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Case of $(0,0)$:
This solution corresponds to the extinction of both species.

The approximating linear system is

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0.75 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

The eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 0.75$.

The critical point $(0,0)$ is an unstable node of both the linear and the nonlinear systems.

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The general (vector) solution of the linear system is

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{0.75t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\mathbf{J(x,y)} = \begin{pmatrix} 1-2x-y & -x \\ -0.5y & 0.75-2y-0.5x \end{pmatrix}$$

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$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{0.75t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

If $c_2 \neq 0$:

$$\begin{pmatrix} x \\ y \end{pmatrix} \sim c_2 e^{0.75t}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad (t \to -\infty)$$

So: in the neighborhood of the origin all trajectories are tangent to the y-axis, except for one trajectory that lies on the x-axis.

$$\mathbf{J(x, y)} = \begin{pmatrix} 1 - 2x - y & -x \\ -0.5y & 0.75 - 2y - 0.5x \end{pmatrix}$$

$$\mathbf{J(x,y)} = \begin{pmatrix} 1-2x-y & -x \\ -0.5y & 0.75-2y-0.5x \end{pmatrix}$$

Case of $(0, 0.75)$**:** In this case, the species x is extinct and y survives.

$$\mathbf{J}(\mathbf{x},\mathbf{y}) = \begin{pmatrix} 1-2x-y & -x \\ -0.5y & 0.75-2y-0.5x \end{pmatrix}$$

Case of $(0, 0.75)$**:** In this case, the species x is extinct and y survives.

The approximate linear system near this critical point is:

$$\frac{d}{dt}\begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} 0.25 & 0 \\ -0.375 & -0.75 \end{pmatrix}\begin{pmatrix} u \\ w \end{pmatrix}$$

where $u = x$ and $w = y - 3/4$.

The eigenvalues are $\lambda_1 = 0.25$ and $\lambda_2 = -0.75$.

The critical point $(0, 0.75)$ is a saddle point and therefore it is unstable for both systems.

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The eigenvalues are $\lambda_1 = 0.25$ and $\lambda_2 = -0.75$.

The critical point $(0, 0.75)$ is a saddle point and therefore it is unstable for both systems.

The general (vector) solution of the linear system is

$$\begin{pmatrix} u \\ w \end{pmatrix} = c_1 e^{0.25t}\begin{pmatrix} 8 \\ -3 \end{pmatrix} + c_2 e^{-0.75t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

If $c_1 = 0$, then there is one pair of trajectories (corresponding to $c_2 e^{-0.75t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$) approaching the critical point along the y-axis for $t \to \infty$.

If $c_2 = 0$, , then there is one pair of trajectories (corresponding to $c_1 e^{0.25t}\begin{pmatrix} 8 \\ -3 \end{pmatrix}$) departing tangent to the line with slope $-3/8$.

$$\mathbf{J}(\mathbf{x},\mathbf{y}) = \begin{pmatrix} 1-2x-y & -x \\ -0.5y & 0.75-2y-0.5x \end{pmatrix}$$

$$\mathbf{J(x,y)} = \begin{pmatrix} 1-2x-y & -x \\ -0.5y & 0.75-2y-0.5x \end{pmatrix}$$

Case of $(1,0)$: In this case, the species x survives and the species y does not.

It is similar to the previous case.

The approximating linear system is

$$\frac{d}{dt}\begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 0 & 0.25 \end{pmatrix}\begin{pmatrix} u \\ w \end{pmatrix}$$

where $u = x - 1$ and $w = y$.

$$\mathbf{J(x,y)} = \begin{pmatrix} 1-2x-y & -x \\ -0.5y & 0.75-2y-0.5x \end{pmatrix}$$

Case of $(1,0)$**:** In this case, the species x survives and the species y does not.

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The eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 0.25$.

The critical point $(1,0)$ is a saddle point and it is unstable for both systems.

$$\mathbf{J(x, y)} = \begin{pmatrix} 1 - 2x - y & -x \\ -0.5y & 0.75 - 2y - 0.5x \end{pmatrix}$$

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The eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 0.25$.

The critical point $(1, 0)$ is a saddle point and it is unstable for both systems.

The general (vector) solution of the linear system is

$$\begin{pmatrix} u \\ w \end{pmatrix} = c_1 e^{-t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{0.25t}\begin{pmatrix} 4 \\ -5 \end{pmatrix}$$

$$\mathbf{J(x,y)} = \begin{pmatrix} 1-2x-y & -x \\ -0.5y & 0.75-2y-0.5x \end{pmatrix}$$

Case of $(1,0)$: In this case, the species x survives and the species y does not.

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The approximating linear system is

$$\frac{d}{dt}\begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 0 & 0.25 \end{pmatrix}\begin{pmatrix} u \\ w \end{pmatrix}$$

where $u = x - 1$ and $w = y$.

The eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 0.25$.

The critical point $(1,0)$ is a saddle point and it is unstable for both systems.

The general (vector) solution of the linear system is

$$\begin{pmatrix} u \\ w \end{pmatrix} = c_1 e^{-t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{0.25t}\begin{pmatrix} 4 \\ -5 \end{pmatrix}$$

One pair of trajectories approaches the critical point along the x-axis, another leaves the critical point tangent to the line of slope $-5/4$. All other trajectories depart from a neighborhood of $(1,0)$.

$$\mathbf{J(x, y)} = \begin{pmatrix} 1 - 2x - y & -x \\ -0.5y & 0.75 - 2y - 0.5x \end{pmatrix}$$

$$\mathbf{J(x,y)} = \begin{pmatrix} 1-2x-y & -x \\ -0.5y & 0.75-2y-0.5x \end{pmatrix}$$

Case of $(0.5, 0.5)$**:** This critical point corresponds to the coexistence of both species.

$$\mathbf{J(x,y)} = \begin{pmatrix} 1-2x-y & -x \\ -0.5y & 0.75-2y-0.5x \end{pmatrix}$$

Case of $(0.5, 0.5)$**:** This critical point corresponds to the coexistence of both species. The approximating linear system is

$$\frac{d}{dt}\begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} -0.5 & -0.5 \\ -0.25 & -0.5 \end{pmatrix}\begin{pmatrix} u \\ w \end{pmatrix}$$

where $u = x - 0.5$ and $w = y - 0.5$.

$$\mathbf{J}(\mathbf{x}, \mathbf{y}) = \begin{pmatrix} 1 - 2x - y & -x \\ -0.5y & 0.75 - 2y - 0.5x \end{pmatrix}$$

Case of $(0.5, 0.5)$**:** This critical point corresponds to the coexistence of both species. The approximating linear system is

$$\frac{d}{dt}\begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} -0.5 & -0.5 \\ -0.25 & -0.5 \end{pmatrix}\begin{pmatrix} u \\ w \end{pmatrix}$$

where $u = x - 0.5$ and $w = y - 0.5$.

The eigenvalues are $\lambda_1 = \frac{-2+\sqrt{2}}{4}$ and $\lambda_2 = \frac{-2-\sqrt{2}}{4}$.

The critical point $(0.5, 0.5)$ is an asymptotically stable node for both systems.

$$\mathbf{J(x,y)} = \begin{pmatrix} 1-2x-y & -x \\ -0.5y & 0.75-2y-0.5x \end{pmatrix}$$

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$$\frac{d}{dt}\begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} -0.5 & -0.5 \\ -0.25 & -0.5 \end{pmatrix}\begin{pmatrix} u \\ w \end{pmatrix}$$

where $u = x - 0.5$ and $w = y - 0.5$.

The eigenvalues are $\lambda_1 = \frac{-2+\sqrt{2}}{4}$ and $\lambda_2 = \frac{-2-\sqrt{2}}{4}$.

The critical point $(0.5, 0.5)$ is an asymptotically stable node for both systems.

The general (vector) solution of the linear system is:

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{(\frac{-2+\sqrt{2}}{4})t}\begin{pmatrix} \sqrt{2} \\ -1 \end{pmatrix} + c_2 e^{(\frac{-2-\sqrt{2}}{4})t}\begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix}$$

All trajectories approach the critical point as $t \to +\infty$. One pair of trajectories approaches the critical point along the line with slope $\sqrt{2}/2$, all others along the line with slope $-\sqrt{2}/2$.

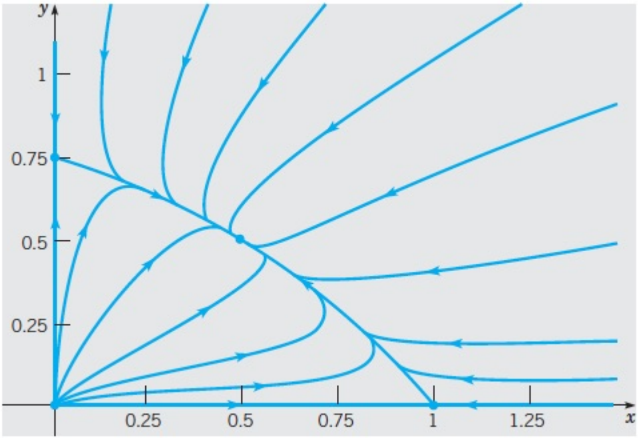

FIGURE 7.3.2 A phase portrait of the system (3).