Section 7.4: Predator-prey equations

Main Topics:

- **Mathematical models of predator-prey situations:**
  One species (the predators) lives on the the other species (the prey), the preys live on a different source of food.

Section 7.4: Predator-prey equations

Main Topics:

- **Mathematical models of predator-prey situations:**
  One species (the predators) lives on the the other species (the prey), the preys live on a different source of food.

  Examples:
  - ▷ Foxes live on the rabbits they prey, while the rabbits live on vegetables.
  - ▷ Lions hunt zebras, while zebras eat grass.

Section 7.4: Predator-prey equations

Main Topics:

- **Mathematical models of predator-prey situations:**
  One species (the predators) lives on the the other species (the prey), the preys live on a different source of food.

  Examples:
  - ▷ Foxes live on the rabbits they prey, while the rabbits live on vegetables.
  - ▷ Lions hunt zebras, while zebras eat grass.
- **Lokta-Volterra equations,**

Section 7.4: Predator-prey equations

Main Topics:

- **Mathematical models of predator-prey situations:**
  One species (the predators) lives on the the other species (the prey), the preys live on a different source of food.

  Examples:
  - ▷ Foxes live on the rabbits they prey, while the rabbits live on vegetables.
  - ▷ Lions hunt zebras, while zebras eat grass.
- **Lokta-Volterra equations,**
- **Long time behavior of solutions.**

The Lotka-Volterra equations

Let $x(t)$ denote the size of the prey and $y(t)$ the size of the predators at time t .

We consider a model for the interaction predator-prey satisfying the following assumptions:

- in the absence of predators, the prey grows at a rate proportional to the current population:

$$\frac{dx}{dt} = ax \quad \text{if } y = 0$$

- in the absence of preys, the predator dies out:

$$\frac{dy}{dt} = -cy, \quad \text{where } c > 0, \text{ if } x = 0$$

The Lotka-Volterra equations

Let $x(t)$ denote the size of the prey and $y(t)$ the size of the predators at time t .

We consider a model for the interaction predator-prey satisfying the following assumptions:

- in the absence of predators, the prey grows at a rate proportional to the current population:
$$\frac{dx}{dt} = ax \quad \text{if } y = 0$$
- in the absence of preys, the predator dies out:
$$\frac{dy}{dt} = -cy, \quad \text{where } c > 0, \text{ if } x = 0$$
- The encounter between the two species promotes the growth of the predators and causes a shrinking of the prey. This means:

  the growth of x is affected by a term $-\alpha xy$, where $\alpha > 0$,

  the growth of y is affected by a term γxy, where $\gamma > 0$.

The equations in the resulting models are known as the **Lotka-Volterra equations**:

$$\frac{dx}{dt} = x(a - \alpha y)$$
$$\frac{dy}{dt} = y(-c + \gamma x)$$

The constants a, b, α and γ are all positive.

- a is the **growth rate** of the prey,
- c is the **death rate** of the predator,
- α and γ measure the **interactions** between the two species.

Example:

$$\frac{dx}{dt} = x(1 - 0.5y)$$

$$\frac{dy}{dt} = y(-0.75 + 0.25x)$$

for x and y positive.

Example:

$$\frac{dx}{dt} = x(1 - 0.5y)$$
$$\frac{dy}{dt} = y(-0.75 + 0.25x)$$

for x and y positive.

Critical points: $(0, 0)$ and $(3, 2)$
The system is almost linear near each critical point because

$$F(x, y) = x(1 - 0.5y) \quad \text{and} \quad G(x, y) = y(-0.75 + 0.25x)$$

have continuous partial derivatives up to order 2 (in fact: of every order).

Example:

$$\frac{dx}{dt} = x(1 - 0.5y)$$
$$\frac{dy}{dt} = y(-0.75 + 0.25x)$$

for x and y positive.

Critical points: $(0, 0)$ and $(3, 2)$

The system is almost linear near each critical point because

$$F(x, y) = x(1 - 0.5y) \quad \text{and} \quad G(x, y) = y(-0.75 + 0.25x)$$

have continuous partial derivatives up to order 2 (in fact: of every order).

Jacobian matrix: $\mathbf{J}(x, y) = \begin{pmatrix} 1 - 0.5y & -0.5x \\ 0.25y & -0.75 + 0.25x \end{pmatrix}$

Example:

$$\frac{dx}{dt} = x(1 - 0.5y)$$
$$\frac{dy}{dt} = y(-0.75 + 0.25x)$$

for x and y positive.

Critical points: $(0, 0)$ and $(3, 2)$
The system is almost linear near each critical point because

$$F(x, y) = x(1 - 0.5y) \quad \text{and} \quad G(x, y) = y(-0.75 + 0.25x)$$

have continuous partial derivatives up to order 2 (in fact: of every order).

Jacobian matrix: $\mathbf{J}(x, y) = \begin{pmatrix} 1 - 0.5y & -0.5x \\ 0.25y & -0.75 + 0.25x \end{pmatrix}$

- **Case of** $(0, 0)$**:** Extinction of both predators and prey.

The approximating linear system is

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -0.75 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Example:

$$\frac{dx}{dt} = x(1 - 0.5y)$$
$$\frac{dy}{dt} = y(-0.75 + 0.25x)$$

for x and y positive.

Critical points: $(0, 0)$ and $(3, 2)$
The system is almost linear near each critical point because

$$F(x, y) = x(1 - 0.5y) \quad \text{and} \quad G(x, y) = y(-0.75 + 0.25x)$$

have continuous partial derivatives up to order 2 (in fact: of every order).

Jacobian matrix: $\mathbf{J}(x, y) = \begin{pmatrix} 1 - 0.5y & -0.5x \\ 0.25y & -0.75 + 0.25x \end{pmatrix}$

- **Case of** $(0, 0)$**:** Extinction of both predators and prey.

The approximating linear system is

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -0.75 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Eigenvalues of the matrix of coefficients: $\lambda_1 = 1$ and $\lambda_2 = -0.75$.

Example:

$$\frac{dx}{dt} = x(1 - 0.5y)$$
$$\frac{dy}{dt} = y(-0.75 + 0.25x)$$

for x and y positive.

Critical points: $(0, 0)$ and $(3, 2)$
The system is almost linear near each critical point because

$$F(x, y) = x(1 - 0.5y) \quad \text{and} \quad G(x, y) = y(-0.75 + 0.25x)$$

have continuous partial derivatives up to order 2 (in fact: of every order).

Jacobian matrix: $\mathbf{J}(x, y) = \begin{pmatrix} 1 - 0.5y & -0.5x \\ 0.25y & -0.75 + 0.25x \end{pmatrix}$

- **Case of** $(0, 0)$**:** Extinction of both predators and prey.

The approximating linear system is

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -0.75 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Eigenvalues of the matrix of coefficients: $\lambda_1 = 1$ and $\lambda_2 = -0.75$.
The critical point $(0, 0)$ is an unstable saddle point for both linear and nonlinear systems.

The general (vector) solution of the linear approximation is

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{-0.75t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

One pair of trajectories approaches the critical point along y-axis, another departs along x-axis. All trajectories different from those along the y-axis depart from the origin.

The general (vector) solution of the linear approximation is

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{-0.75t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

One pair of trajectories approaches the critical point along y-axis, another departs along x-axis. All trajectories different from those along the y-axis depart from the origin.

- **Case of** $(3, 2)$**:** Survival of both predators and prey.

The general (vector) solution of the linear approximation is

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{-0.75t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

One pair of trajectories approaches the critical point along y-axis, another departs along x-axis. All trajectories different from those along the y-axis depart from the origin.

- **Case of** $(3, 2)$**:** Survival of both predators and prey.

Recall: $\mathbf{J}(x, y) = \begin{pmatrix} 1 - 0.5y & -0.5x \\ 0.25y & -0.75 + 0.25x \end{pmatrix}$

The general (vector) solution of the linear approximation is

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{-0.75t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

One pair of trajectories approaches the critical point along y-axis, another departs along x-axis. All trajectories different from those along the y-axis depart from the origin.

- **Case of** $(3, 2)$**:** Survival of both predators and prey.

Recall: $\mathbf{J}(x, y) = \begin{pmatrix} 1 - 0.5y & -0.5x \\ 0.25y & -0.75 + 0.25x \end{pmatrix}$

The approximating linear system is

$$\frac{d}{dt}\begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} 0 & -1.5 \\ 0.5 & 0 \end{pmatrix}\begin{pmatrix} u \\ w \end{pmatrix} \quad \text{i.e.} \quad \frac{d}{dt}\begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} 0 & -3/2 \\ 1/2 & 0 \end{pmatrix}\begin{pmatrix} u \\ w \end{pmatrix}$$

where $u = x - 3$ and $w = y - 2$ and the matrix is $\mathbf{J}(3, 2)$.
The eigenvalues of $\mathbf{J}(3, 2)$ are purely imaginary:

$$\lambda_1 = i\sqrt{3}/2 \quad \text{and} \quad \lambda_2 = -i\sqrt{3}/2\,.$$

The critical point $(3, 2)$ is a stable center for the above linear differential system.

What about the nonlinear system?

The general (vector) solution of the linear approximation is

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{-0.75t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

One pair of trajectories approaches the critical point along y-axis, another departs along x-axis. All trajectories different from those along the y-axis depart from the origin.

- **Case of** $(3, 2)$**:** Survival of both predators and prey.

Recall: $\mathbf{J}(x, y) = \begin{pmatrix} 1 - 0.5y & -0.5x \\ 0.25y & -0.75 + 0.25x \end{pmatrix}$

The approximating linear system is

$$\frac{d}{dt}\begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} 0 & -1.5 \\ 0.5 & 0 \end{pmatrix}\begin{pmatrix} u \\ w \end{pmatrix} \quad \text{i.e.} \quad \frac{d}{dt}\begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} 0 & -3/2 \\ 1/2 & 0 \end{pmatrix}\begin{pmatrix} u \\ w \end{pmatrix}$$

where $u = x - 3$ and $w = y - 2$ and the matrix is $\mathbf{J}(3, 2)$.
The eigenvalues of $\mathbf{J}(3, 2)$ are purely imaginary:

$$\lambda_1 = i\sqrt{3}/2 \quad \text{and} \quad \lambda_2 = -i\sqrt{3}/2\,.$$

The critical point $(3, 2)$ is a stable center for the above linear differential system.

What about the nonlinear system?

The table of Section 7.2 does not provide an answer in this case.

Idea (still for the linear system): find a relation between u and w.
The linear equations

$$\frac{du}{dt} = -\frac{3}{2}w \quad \text{and} \quad \frac{dw}{dt} = \frac{1}{2}u$$

imply

$$\frac{dw}{du} = \frac{dw/dt}{du/dt} = \frac{\frac{1}{2}u}{-\frac{3}{2}w} = -\frac{1}{3}\frac{u}{w}$$

i.e.

$$3w\,\frac{dw}{du} = -u$$

Idea (still for the linear system): find a relation between u and w.
The linear equations

$$\frac{du}{dt} = -\frac{3}{2}w \quad \text{and} \quad \frac{dw}{dt} = \frac{1}{2}u$$

imply

$$\frac{dw}{du} = \frac{dw/dt}{du/dt} = \frac{\frac{1}{2}u}{-\frac{3}{2}w} = -\frac{1}{3}\frac{u}{w}$$

i.e.

$$3w\,\frac{dw}{du} = -u$$

This is a separable first-order differential equation.
By integration

$$u^2 + 3w^2 = C\,, \qquad C\text{=constant}$$

Concentric ellipses with center $(0,0)$ in the (u,v) plane,

i.e. concentric ellipses with center $(3,2)$ in the (x,y) plane.

Try the same method for the nonlinear system:
We know that:

$$\frac{dx}{dt} = x(1 - 0.5y)$$
$$\frac{dy}{dt} = y(-0.75 + 0.25x)$$

So

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-0.75 + 0.25x}{x} \frac{y}{1 - 0.5y}$$

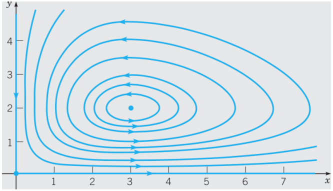
i.e.

$$\left(\frac{1}{y} - \frac{1}{2}\right)\frac{dy}{dx} = -\frac{3}{4}\frac{1}{x} + \frac{1}{4}$$

By integration:
$\frac{3}{4}\ln x + \ln y - \frac{1}{2}y - \frac{1}{4} = C$, C=constant

The trajectories are closed curves around the critical point.

The critical point is a center and is stable.
The evolution of the predator-prey system is cyclic.



Study of general Lotka-Volterra equations

$$\frac{dx}{dt} = x(a - \alpha y) \qquad \longleftarrow F(x,y)$$
$$\frac{dy}{dt} = y(-c + \gamma x) \qquad \longleftarrow G(x,y)$$

The critical points are $(0,0)$ and $(c/\gamma, a/\alpha)$.
The system is almost linear near each of its critical points.

Case of $(0,0)$**:** Extinction of both species.

The approximating linear system is

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & -c \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Eigenvalues are $\lambda_1 = a$, $\lambda_2 = -c$.
So $(0,0)$ is a saddle point (unstable) for both the linear and the nonlinear systems.

Case of $(c/\gamma, a/\alpha)$**:** survival of both species.

The approximating linear system is

$$\frac{d}{dt}\begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} 0 & -\alpha c/\gamma \\ \gamma a/\alpha & 0 \end{pmatrix} \begin{pmatrix} u \\ w \end{pmatrix}$$

with $u = x - c/\gamma$ and $w = y - a/\alpha$.
The eigenvalues are $\lambda_1 = i\sqrt{ac}$ and $\lambda_2 = -i\sqrt{ac}$.
So $(c/\gamma, a/\alpha)$ is a stable center for the linear system.

To determine the trajectories of the linear approximation:

$$\frac{dw}{du} = \frac{dw/dt}{du/dt} = -\frac{(\gamma a/\alpha)u}{(\alpha c/\gamma)w},$$

which can be rewritten as:

$$\gamma^2 au\frac{dw}{du} = -\alpha^2 cw$$

. Thus:

$$\gamma^2 au^2 + \alpha^2 cw^2 = k, \qquad k = \text{constant} \geq 0\,.$$

The trajectoires are concentric ellipses centered at the critical point.

For the nonlinear system, one can proceed as in the example:

We know that:

$$\frac{dx}{dt} = x(a - \alpha y) \qquad \leftarrow F(x, y)$$

$$\frac{dy}{dt} = y(-c + \gamma x) \qquad \leftarrow G(x, y)$$

So

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{(-c + \gamma x)}{x} \, \frac{y}{(a - \alpha y)}$$

which is a separable differential equation.

By integration one gets the implicit solution

$$a \ln y - \alpha y + c \ln x - \gamma x = C$$

where C is a constant.

The graph of a trajectory corresponding to each fixed value of C is a closed curve surrounding the critical point $(c/\gamma, a/\alpha)$. This critical point is a center.

The predator-prey system presents a cyclic variation.

.