Graphical Methods – for linear systems: Sections 3.2, 3.3, 3.4, 3.5

Main topics:

- Component plots
- Autonomous systems (here linear systems only):
  - ◇ trajectories (or orbits)
  - ◇ equilibrium solutions
  - ◇ direction fields
  - ◇ phase portraits.
- For homogenous linear systems with constant coeffcient: $\mathbf{x}' = \mathbf{A}\mathbf{x}$:
  phase portraits and stability, according to the nature of the eigenvalues of $\mathbf{A}$.
  Classification of the equilibrium solutions.
- The case of non-homogenous systems $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{b}$ with $\mathbf{A}$ invertible.

Plotting solutions

Definition

Let $\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ be a solution of the IVP

$$\mathbf{x}'(t) = \mathbf{P}(t)\mathbf{x}(t) + \mathbf{g}(t) \quad \text{with initial condition} \quad \mathbf{x}_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

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More effective representations for *autonomous systems* (see next slide):

- trajectories (or orbits)
- direction fields
- phase portraits

Autonomous systems of two linear 1st order DE's

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A system of two linear DE's $\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t)$ is **autonomous** if $\mathbf{P}$ and $\mathbf{g}$ are constant in t, i.e. it is of the form

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{b}$$

where:

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Remarks:

- In Chapter 1, a first-order DE was said to be **autonomous** if of the form $\frac{dy}{dt} = f(y)$, where f is constant in t.
  For a *linear* DE, this means that $\frac{dy}{dt} = \alpha y + \beta$, where α, β are real numbers.
- In Section 3.6, we will give the general definition of autonomous systems of first order DE.
  For linear systems, "autonomous" just means "with constant coefficients".

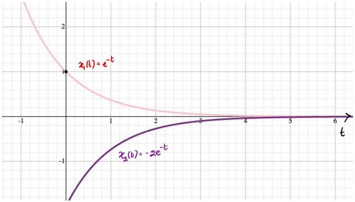
Consider the autonomous (=constant coefficient) system of two 1st order linear DE's:

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{b} \quad \text{where} \quad \mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}.$$

- $\mathbf{x}$ is called the **state vector**.
- x_1 and x_2 are called the **state variables**.

Example:

One can check that $\mathbf{x}(t) = e^{-t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ is a solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ where $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$.
Hence $x_1(t) = e^{-t}$ and $x_2(t) = -2e^{-t}$.
Component plots of the solution:



- The x_1x_2-plane is called the **phase plane** (or **state plane)**.
- Let $x_1 = x_1(t), x_2 = x_2(t)$ be a solution. The curve $t \mapsto (x_1(t), x_2(t))$ in the phase plane is a **trajectory** (or **orbit**).

Example (continued):

For $\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = e^{-t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} e^{-t} \\ -2e^{-t} \end{pmatrix}$:

We elimitate the t-variable:

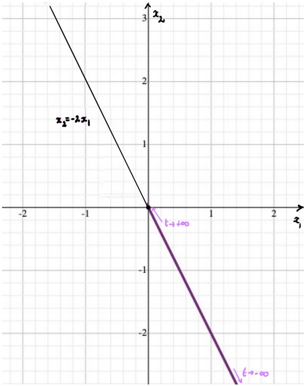
$$\frac{x_2}{x_1} = \frac{-2e^{-t}}{e^{-t}} = -2 \quad \text{and get} \quad x_2 = -2x_1 \quad \text{(a line)}\,.$$

The trajectory lies on this line, but it is not the entire line.

Since

$$\lim_{t\to-\infty} e^{-t} = +\infty \quad \text{and} \quad \lim_{t\to+\infty} e^{-t} = 0$$

it is the violet half-line in the x_1x_2-plane (i.e. the phase plane)



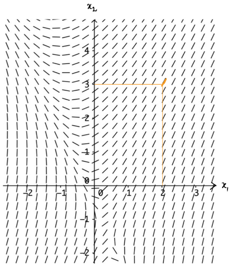
Remark: this method allows us to draw the trajectory of any solution of the form $e^{\lambda t}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$.

- A **direction field** is an array of vectors in the phase space: a vector parallel to $\mathbf{Ax} + \mathbf{b}$ is drawn with its tail at $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ for every choice of (x_1, x_2) in a fixed grid. The lenght of the vectors is often chosen to be constant.
- For readability, on a dense grid, one often draws a segment instead of a vector.
- If a trajectory passes through a point (x_1, x_2) of the grid, then its tangent vector at (x_1, x_2) is a (multiple of a) vector of the direction field. Conversely, we can use a direction field to “guess" trajectories.

Example: $\begin{cases} \dfrac{dx_1}{dt} = x_1 + x_2 + 1 \\ \dfrac{dx_2}{dt} = 4x_1 + x_2 \end{cases}$ with $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

At the point (x_1, x_2) we draw a small vector (often just a segment on a dense grid) of slope $\frac{4x_1+x_2}{x_1+x_2+1}$ with same direction as $(x_1', x_2') = (x_1 + x_2 + 1, 4x_1 + x_2)$.

e.g. at the point $(2, 3)$ we draw a small segment of slope $\frac{4 \cdot 2+3}{2+3+1} = \frac{11}{6}$, directed as $(6, 11)$.



Direction field in the phase plane $x_1 x_2$

- An **equilibrium point** is a solution for which $\frac{d\mathbf{x}}{dt} = 0$, i.e. $\mathbf{Ax} + \mathbf{b} = 0$.

  Equilibrium solutions are also called **equilibrium solutions** or **critical points**.

  ▷ If the matrix $\mathbf{A}$ is non singular (i.e. $\det(\mathbf{A}) \neq 0$), there is a unique equilibrium solution given by $\mathbf{x} = -\mathbf{A}^{-1}\mathbf{b}$. The equilibrium solution is a point $\mathbf{x}_{\mathrm{eq}}$ in the phase plane.

  ▷ If the matrix $\mathbf{A}$ is singular (i.e. $\det(\mathbf{A}) = 0$), there is either infinitely many equilibrium solutions or no equilibrium solution at all.

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$\det(\mathbf{A}) \neq 0$. Unique equilibrium point = solution of $\begin{cases} x_1 + x_2 + 1 = 0 \\ 4x_1 + x_2 = 0 \end{cases}$, i.e. $\mathbf{x}_{eq} = \begin{pmatrix} 1/3 \\ -4/3 \end{pmatrix}$.

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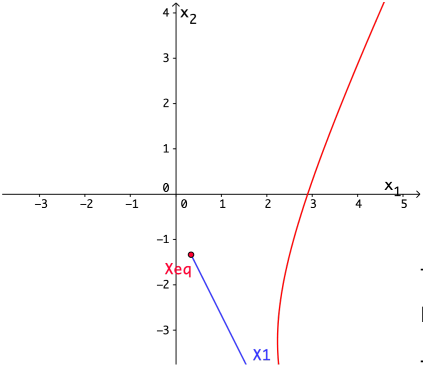
Example: $\begin{cases} \frac{dx_1}{dt} = x_1 + 1 \\ \frac{dx_2}{dt} = 4x_1 \end{cases}$ with $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 4 & 0 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\det(\mathbf{A}) = 0$ and the system $\begin{cases} x_1 + 1 = 0 \\ 4x_1 = 0 \end{cases}$ has no solution. There are no equilibrium solutions.

- A **phase portrait** is the plot a of representative sample of trajectories, including the equilibrium points, in the phase plane.

Example:

$$\begin{cases} \dfrac{dx_1}{dt} = x_1 + x_2 + 1 \\ \dfrac{dx_2}{dt} = 4x_1 + x_2 \end{cases} \quad \text{with} \quad \mathbf{A} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$



The phase plane $x_1 x_2$.

Equilibrium solution $\mathbf{x}_{eq} = \begin{pmatrix} 1/3 \\ -4/3 \end{pmatrix}$

Trajectory for $\mathbf{x}_1(t) = \begin{pmatrix} e^{-t} + 1/3 \\ -2e^{-t} - 4/3 \end{pmatrix}$

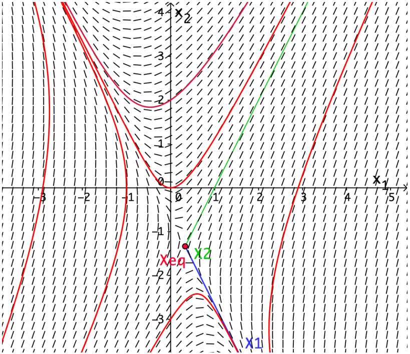
(with $\lim_{t \to +\infty} \mathbf{x}_1(t) = \mathbf{x}_{eq}$)

A third trajectory in red.

Example (continued):

Phase portrait with the direction field, X_{eq}, and a few trajectories for the same example:

$$\begin{cases} \dfrac{dx_1}{dt} = x_1 + x_2 + 1 \\ \dfrac{dx_2}{dt} = 4x_1 + x_2 \end{cases} \quad \text{with} \quad \mathbf{A} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$



Phase portraits and stability for homogeneous systems $\mathbf{x}' = \mathbf{A}\mathbf{x}$ (constant coefficient matrix $\mathbf{A}$)

We shall suppose in the following that $\mathbf{A}$ is not the zero matrix.

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We shall suppose in the following that **A** is not the zero matrix.

The form of the general solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ depends on the properties (real, complex, distinct, nonzero...) of the eigenvalues λ_1, λ_2 of **A**.

Likewise, the stability properties of the solutions of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ depend on λ_1, λ_2 of **A**.

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Likewise, the stability properties of the solutions of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ depend on λ_1, λ_2 of $\mathbf{A}$.

Equilibrium points

For equilibrium (or critical) points $\mathbf{x_{eq}}$ (=solutions of $\mathbf{A}\mathbf{x} = 0$), there are two main cases:

- Both λ_1, λ_2 are non-zero, i.e. $\det(\mathbf{A}) \neq 0$. Then $\mathbf{x_{eq}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is the unique equilibrium point.
- One of λ_1, λ_2 is zero, i.e. $\det(\mathbf{A}) = 0$, then there is one straight line of equilibrium points $\mathbf{x_{eq}}$.

When $\det(\mathbf{A}) \neq 0$ the unique equilibrium point $\mathbf{x}_{\mathbf{eq}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is classified into different types and stability properties:

TABLE 3.5.2 Stability properties of linear systems $\mathbf{x}' = \mathbf{A}\mathbf{x}$ with $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ and $\det \mathbf{A} \neq 0$.

$\det(A) \neq 0 \Leftrightarrow \lambda_1 \neq 0, \lambda_2 \neq 0$

	Eigenvalues of A		Type of Critical Point	Stability
real and distinct eigenvalues	$\lambda_1 > \lambda_2 > 0$	equal sign	Node	Unstable
	$\lambda_1 < \lambda_2 < 0$		Node	Asymptotically stable
	$\lambda_2 < 0 < \lambda_1$	opposite sign	Saddle point	Unstable
real and coincident eigenvalues	$\lambda_1 = \lambda_2 > 0$		Proper or improper node	Unstable
	$\lambda_1 = \lambda_2 < 0$		Proper or improper node	Asymptotically stable
complex conjugate eigenvalues	$\lambda_1, \lambda_2 = \mu \pm i\nu$		Spiral point	
	$\mu > 0$			Unstable
	$\mu < 0$			Asymptotically stable
	$\lambda_1 = i\nu, \lambda_2 = -i\nu$	purely imaginary	Center	Stable

J. BRENNAN & W. BOYCE, DIFFERENTIAL EQUATIONS, WILEY

The precise definitions of these terms will be given in Section 7.1.

The geometry of the trajectories in the phase plane we are going to describe in the following explain their basic meaning.

I. A with real, distinct, nonzero eigenvalues $\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_1 \neq \lambda_2$

The **general solution** of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is

$$\mathbf{x}(t) = C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2 \qquad C_1, C_2 \in \mathbb{R}$$

where:

- $\mathbf{v}_1 = \begin{pmatrix} v_{11} \\ v_{21} \end{pmatrix}$ is an eigenvector of **A** of eigenvalue λ_1
- $\mathbf{v}_2 = \begin{pmatrix} v_{12} \\ v_{22} \end{pmatrix}$ is an eigenvector of **A** of eigenvalue λ_2

(Recall that $\mathbf{v}_1$ and $\mathbf{v}_1$ are linearly independent because $\lambda_1 \neq \lambda_2$).

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Example 1:

Consider the system of two linear DE's $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix}$.

Its general solution is $\mathbf{x}(t) = C_1 e^{-3t} \begin{pmatrix} 1 \\ -3 \end{pmatrix} + C_2 e^{-2t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

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- Compute $\lim_{t\to+\infty} \mathbf{x}(t)$. [The limit is $\mathbf{x}_{\text{eq}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ because $\lim_{t\to\infty} e^{-at} = 0$ if $a > 0$.]
- Show that, if $C_2 \neq 0$, then

$$\mathbf{x}(t) \quad \sim_{t\to+\infty} \quad C_2 e^{-2t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

[*Answer:* If $C_2 \neq 0$, then

$$\mathbf{x}(t) = C_1 e^{-3t} \begin{pmatrix} 1 \\ -3 \end{pmatrix} + C_2 e^{-2t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = e^{-2t} \left[C_1 e^{-t} \begin{pmatrix} 1 \\ -3 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right] \sim_{t \to +\infty} C_2 e^{-2t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

- Show that, if $C_1 \neq 0$, then

$$\mathbf{x}(t) \quad \sim_{t \to -\infty} \quad C_1 e^{-3t} \begin{pmatrix} 1 \\ -3 \end{pmatrix} .$$

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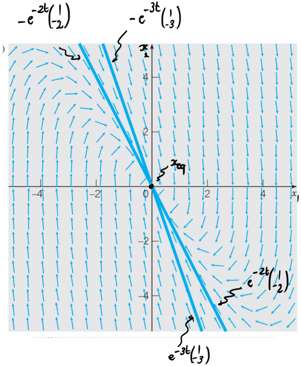


Interpretation:

The trajectories of the solutions which are not parallel to $e^{-3t}\begin{pmatrix}1\\-3\end{pmatrix}$ (i.e. if $C_2 \neq 0$) move towards $\mathbf{x}_{\text{eq}}$ for $t \to +\infty$ along directions parallel to $\begin{pmatrix}1\\-2\end{pmatrix}$

The trajectories of the solutions which are not parallel to $\begin{pmatrix}1\\-2\end{pmatrix}$ (i.e. if $C_1 \neq 0$) arrive from $t \to -\infty$ along directions parallel to $\begin{pmatrix}1\\-3\end{pmatrix}$

▷ $\lambda_1 \neq \lambda_2$ **and both negative,** WLOG $\lambda_1 < \lambda_2 < 0$**:**

Then $\lim_{t\to+\infty} \mathbf{x}(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \mathbf{x_{eq}}$, i.e. all trajectories approach $\mathbf{x_{eq}}$ asymptotically as $t \to +\infty$: we say that $\mathbf{x_{eq}}$ is **asymptotically stable** and we call it a **nodal sink**.

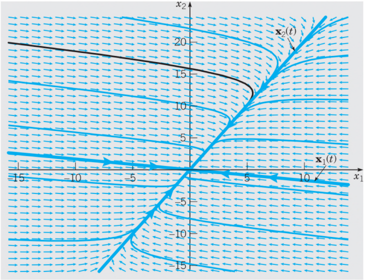
If $C_2 \neq 0$, then
$$\mathbf{x}(t) = e^{\lambda_2 t}\left(C_1 e^{(\lambda_1-\lambda_2)t}\mathbf{v}_1 + C_2\mathbf{v}_2\right) \sim_{t\to+\infty} C_2 e^{\lambda_2 t}\mathbf{v}_2 = C_2\mathbf{x}_2(t)$$

i.e. all solutions with $C_2 \neq 0$ approach $\mathbf{x_{eq}}$ along the direction of $C_2\mathbf{v}_2$ (=the eigenvector with eigenvalue closest to 0).

If $C_1 \neq 0$, then
$$\mathbf{x}(t) = e^{\lambda_1 t}\left(C_1\mathbf{v}_1 + C_2 e^{(\lambda_2-\lambda_1)t}\mathbf{v}_2\right) \sim_{t\to-\infty} C_1 e^{\lambda_1 t}\mathbf{v}_1 = C_1\mathbf{x}_1(t)$$

i.e. all solutions with $C_1 \neq 0$ backward in time approach the direction of $C_1\mathbf{v}_1$.

(b)



J. BRENNAN et al., DIFFERENTIAL EQUATIONS, FIGURE 3.3.2

▷ $\lambda_1 \neq \lambda_2$ **and both positive, WLOG** $\lambda_1 > \lambda_2 > 0$**:**

Then all solutions diverge from $\mathbf{x_{eq}}$ for $t \to +\infty$ and $\lim_{t\to-\infty} \mathbf{x}(t) = \mathbf{x_{eq}}$, i.e. all trajectories approach $\mathbf{x_{eq}}$ backward in time $t \to -\infty$: we say that $\mathbf{x_{eq}}$ is **unstable** and we call it a **nodal source**.

If $C_2 \neq 0$, then $\quad \mathbf{x}(t) = e^{\lambda_2 t}\left(C_1 e^{(\lambda_1-\lambda_2)t}\mathbf{v}_1 + C_2\mathbf{v}_2\right) \quad \sim_{t\to-\infty} \quad C_2 e^{\lambda_2 t}\mathbf{v}_2 = C_2\mathbf{x}_2(t)$

i.e. backward in time all solutions with $C_2 \neq 0$ approach $\mathbf{x_{eq}}$ along the direction of $C_2\mathbf{v}_2$ (=the eigenvector with eigenvalue closest to 0).

If $C_1 \neq 0$, then $\quad \mathbf{x}(t) = e^{\lambda_1 t}\left(C_1\mathbf{v}_1 + C_2 e^{(\lambda_2-\lambda_1)t}\mathbf{v}_2\right) \quad \sim_{t\to+\infty} \quad C_1 e^{\lambda_1 t}\mathbf{v}_1 = C_1\mathbf{x}_1(t)$

i.e. all solutions with $C_1 \neq 0$ move for $t \to \infty$ asymptotically to a line of direction $C_1\mathbf{v}_1$.

(Picture as in the previous case “with arrows reversed”)

▷ **$\lambda_1 \neq \lambda_2$ of opposite sign, WLOG $\lambda_2 < 0 < \lambda_1$:**

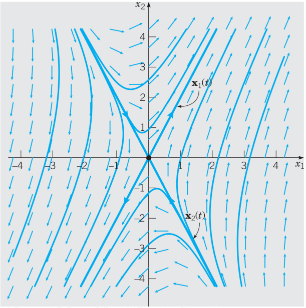
In this case there are solutions that tend to $\mathbf{x_{eq}}$ for $t \to +\infty$, but most of the solutions (those for $C_1 \neq 0$) grows to infinity: indeed, if $C_1 = 0$, then

$$\lim_{t \to +\infty} C_2 e^{\lambda_2 t} \mathbf{v}_2 = \mathbf{x_{eq}}$$

and if $C_1 \neq 0$, then

$$\mathbf{x}(t) = C_1 e^{\lambda_1 t} \mathbf{v}_1 + C_2 e^{\lambda_2 t} \mathbf{v}_2 \sim_{t \to +\infty} C_1 e^{\lambda_1 t} \mathbf{v}_1$$

grow to infinity in the direction of $C_1 \mathbf{v}_1$. We say that $\mathbf{x}_{\mathrm{eq}}$ is **unstable** and we call it a **saddle point**.



J. Brannan & E. Boyce, Differential Equations, Figure 3.3.4

II. A with real, distinct eigenvalues and $\lambda_1 = 0$

The general solution is $\mathbf{x}(t) = C_1\mathbf{x}_1(t) + C_2\mathbf{x}_2(t) = C_1\mathbf{v}_1 + C_2e^{\lambda_2 t}\mathbf{v}_2$

Each eigenvector $\mathbf{v}$ of eigenvalue $\lambda_1 = 0$ is of the form $C_1\mathbf{v}_1$ and satisfies $\mathbf{Av} = 0$. So we have a line ℓ of critical points $\mathbf{x}_{\text{eq}}$.

II. A with real, distinct eigenvalues and $\lambda_1 = 0$

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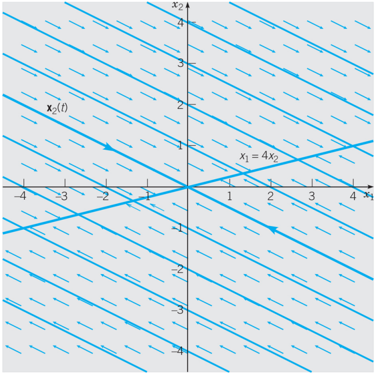
Each eigenvector $\mathbf{v}$ of eigenvalue $\lambda_1 = 0$ is of the form $C_1\mathbf{v}_1$ and satisfies $\mathbf{A}\mathbf{v} = 0$. So we have a line ℓ of critical points $\mathbf{x}_{\text{eq}}$.

J. Brannan & W. Boyce, Differential equations, Figure 3.3.8, $\lambda_1 = 0$, $\lambda_2 < 0$



▷ $\lambda_1 = 0$ **and** $\lambda_2 < 0$

The trajectories $\mathbf{x}(t) = C_1\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2$ passing through points not in ℓ (i.e. with $C_2 \neq 0$) move along half lines parallel to $C_2\mathbf{v}_2$ and asymptotically tend to the point on ℓ given by $\lim_{t\to+\infty} \mathbf{x}(t) = C_1\mathbf{v}_1$.

II. A with real, distinct eigenvalues and $\lambda_1 = 0$

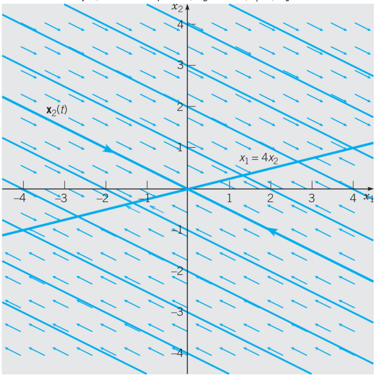
The general solution is $\mathbf{x}(t) = C_1\mathbf{x}_1(t) + C_2\mathbf{x}_2(t) = C_1\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2$

Each eigenvector $\mathbf{v}$ of eigenvalue $\lambda_1 = 0$ is of the form $C_1\mathbf{v}_1$ and satisfies $\mathbf{A}\mathbf{v} = 0$. So we have a line ℓ of critical points $\mathbf{x}_{\text{eq}}$.

▷ $\lambda_1 = 0$ **and** $\lambda_2 < 0$

The trajectories $\mathbf{x}(t) = C_1\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2$ passing through points not in ℓ (i.e. with $C_2 \neq 0$) move along half lines parallel to $C_2\mathbf{v}_2$ and asymptotically tend to the point on ℓ given by $\lim_{t\to+\infty}\mathbf{x}(t) = C_1\mathbf{v}_1$.

J. Brannan & W. Boyce, Differential equations, Figure 3.3.8, $\lambda_1 = 0$, $\lambda_2 < 0$



▷ $\lambda_1 = 0$ **and** $\lambda_2 > 0$

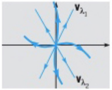
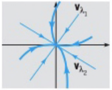
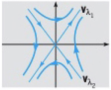
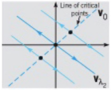
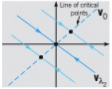
The situation is as above, with direction of the trajectories not passing though the critical line reversed since

$$\lim_{t\to-\infty}\mathbf{x}(t) = C_1\mathbf{v}_1 .$$

i.e. the trajectories passing through points which are not on ℓ tend to points on ℓ backward in time and diverge to infinity for $t \to +\infty$.

Summary of cases I. and II.

TABLE 3.3.1 Phase portraits for $\mathbf{x}' = \mathbf{A}\mathbf{x}$ when $\mathbf{A}$ has distinct real eigenvalues.

	Eigenvalues	Sample Phase Portrait	Type of Critical Point	Stability
I. $(\det(A) \neq 0)$	$\lambda_1 \neq \lambda_2$ Both positive		(0, 0) is a **nodal source**.	Unstable
	$\lambda_1 \neq \lambda_2$ Both negative		(0, 0) is a **nodal sink**.	Asymptotically stable
	$\lambda_1 \neq \lambda_2$ Opposite signs		(0, 0) is a **saddle**.	Unstable
II. $(\det(A) = 0)$	$\lambda_1 = 0$ and $\lambda_2 > 0$		line of equilibrium points; type not defined	
	$\lambda_1 = 0$ and $\lambda_2 < 0$			

III. A with complex conjugate eigenvalues and $\lambda_2 = \overline{\lambda_1} \neq 0$

Write:

- λ instead of λ_1.
  Set $\lambda = \mu + i\nu$ with $\mu, \nu \in \mathbb{R}$.
- $\mathbf{v}$ for a fixed eigenvector of $\mathbf{A}$ for the eigenvalue λ.
  Set $\mathbf{v} = \mathbf{a} + i\mathbf{b}$ with $\mathbf{a}, \mathbf{b}$ real vectors.

General solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$:

$$\mathbf{x}(t) = C_1 \operatorname{Re} \mathbf{x}_1(t) + C_2 \operatorname{Im} \mathbf{x}_1(t)$$

where

$$\mathbf{x}_1(t) = e^{(\mu + i\nu)t}\mathbf{v}$$

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General solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$:

$$\mathbf{x}(t) = C_1 \operatorname{Re} \mathbf{x}_1(t) + C_2 \operatorname{Im} \mathbf{x}_1(t)$$

where

$$\mathbf{x}_1(t) = e^{(\mu+i\nu)t}\mathbf{v}$$

Explicit computations bring the set of fundamental solutions in the form:

$$\operatorname{Re} \mathbf{x}_1(t) = e^{\mu t}[\cos(\nu t)\mathbf{a} - \sin(\nu t)\mathbf{b}]$$
$$\operatorname{Im} \mathbf{x}_1(t) = e^{\mu t}[\cos(\nu t)\mathbf{b} + \sin(\nu t)\mathbf{a}] \,.$$

Since $\det(\mathbf{A}) = \lambda\overline{\lambda} \neq 0$

$$\mathbf{x}_{\text{eq}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

is the unique equilibrium (or critical) point.

The solutions

$$\begin{aligned} \operatorname{Re} \mathbf{x}_1(t) &= e^{\mu t}[\cos(\nu t)\mathbf{a} - \sin(\nu t)\mathbf{b}] \\ \operatorname{Im} \mathbf{x}_1(t) &= e^{\mu t}[\cos(\nu t)\mathbf{b} + \sin(\nu t)\mathbf{a}] \,. \end{aligned}$$

have an oscillatory behavior as functions of t.

The nature of the oscillation depends on $\mu = \operatorname{Re} \lambda$.

Example:

In section 3.5 we found that $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, has general solution

$$\mathbf{x}(t) = C_1 \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + C_2 \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$$

where C_1, C_2 are constants and $t \in \mathbb{R}$. *Remark:* periodic of period 2π.

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Find the solution of the IVP for $\mathbf{x}' = \mathbf{A}\mathbf{x}$ with initial condition $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

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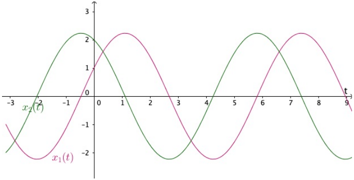
Find the solution of the IVP for $\mathbf{x}' = \mathbf{A}\mathbf{x}$ with initial condition $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

The solution of the IVP is $\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$, where

$$x_1(t) = \cos t + 2\sin t = \sqrt{5}\cos(\theta - t)$$
$$x_2(t) = -\sin t + 2\cos t = \sqrt{5}\sin(\theta - t)$$

and θ is chosen so that $\cos\theta = \frac{1}{\sqrt{5}}$, $\sin\theta = \frac{2}{\sqrt{5}}$



Component plots.

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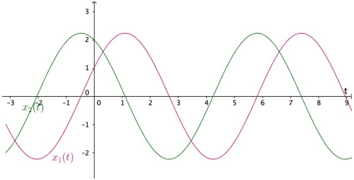
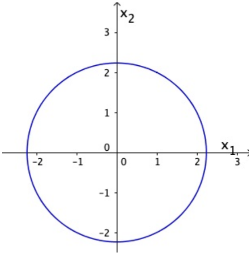


Component plots.



The trajectory of the solution in the phase plane lies on the circle $x_1^2 + x_2^2 = 5$.

Back to the general case, with set of fundamental solutions:

$$\mathrm{Re}\,\mathbf{x}_1(t) = e^{\mu t}[\cos(\nu t)\mathbf{a} - \sin(\nu t)\mathbf{b}]$$
$$\mathrm{Im}\,\mathbf{x}_1(t) = e^{\mu t}[\cos(\nu t)\mathbf{b} + \sin(\nu t)\mathbf{a}]\,.$$

- $\mu = 0$, **i.e.** $\lambda = i\nu$**:** [see previous example, where $\lambda = i$]

$\mathrm{Re}\,\mathbf{x}_1$ and $\mathrm{Im}\,\mathbf{x}_1$ are periodic function of t of period $T = \frac{2\pi}{\nu}$.
In the phase plane:
The trajectories are ellipses.
The origin $(0,0)$ is a called a **center** and it is said to be **stable**.

- $\mu < 0$:

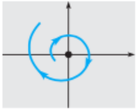
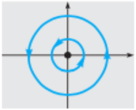
The amplitude of the oscillations of $\mathrm{Re}\,\mathbf{x}_1$ and $\mathrm{Im}\,\mathbf{x}_1$ decays exponentially as $t \to +\infty$.
In the phase plane:
The trajectories spiral around $(0,0)$ and approach $(0,0)$ at $t \to +\infty$.
The origin $(0,0)$ is a called a **spiral sink** and it is said to be **asymptotically stable**.

- $\mu > 0$:

The amplitude of the oscillations of $\mathrm{Re}\,\mathbf{x}_1$ and $\mathrm{Im}\,\mathbf{x}_1$ grows exponentially as $t \to +\infty$.
In the phase plane:
The trajectories spiral out of $(0,0)$ as t increases.
The origin $(0,0)$ is a called a **spiral source** and it is said to be **unstable**.

Summary of case III.

TABLE 3.4.1 Phase portraits for $\mathbf{x}' = \mathbf{A}\mathbf{x}$ when $\mathbf{A}$ has complex eigenvalues.

Eigenvalues	Sample Phase Portrait	Type of Critical Point	Stability
$\lambda = \mu \pm i\nu$ $\mu < 0$		(0, 0) is a **spiral sink**.	Asymptotically stable
$\lambda = \mu \pm i\nu$ $\mu > 0$		(0, 0) is a **spiral source**.	Unstable
$\lambda = \mu \pm i\nu$ $\mu = 0$		(0, 0) is a **center**.	Stable

IV. A with two repeated real eigenvalues equal to λ

$\lambda \neq 0$ because $\mathbf{A} \neq \mathbf{0}$. So $\det(\mathbf{A}) = \lambda^2 \neq 0$.

Thus $\mathbf{x}' = \mathbf{A}\mathbf{x}$ has a unique equilibrium solution $\mathbf{x}_{\text{eq}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

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Recall that two cases have to be distinguished:

- $\mathbf{A} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ is a diagonal matrix.
- $\mathbf{A}$ is not a diagonal matrix.

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We will not treat in details the case where $\mathbf{A}$ is not diagonal.

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We will not treat in details the case where $\mathbf{A}$ is not diagonal.

If $\mathbf{A} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$:

then $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are two linearly indep. eigenvectors with eigenvalue λ.

IV. A with two repeated real eigenvalues equal to λ

$\lambda \neq 0$ because $\mathbf{A} \neq \mathbf{0}$. So $\det(\mathbf{A}) = \lambda^2 \neq 0$.

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The general solution is $\mathbf{x}(t) = C_1 e^{\lambda t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{\lambda t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = e^{\lambda t} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$

IV. A with two repeated real eigenvalues equal to λ

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Recall that two cases have to be distinguished:

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In the phase space, the trajectory of **every** solution is a half-line with endpoint 0.

Example: $[\lambda = -1]$

Let $\mathbf{A} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. Find the solution of the IVP: $\mathbf{x}' = \mathbf{A}\mathbf{x}$ with $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

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As above, the general solution is $\mathbf{x}(t) = e^{-t} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$.

Determine the value of C_1, C_2 from the initial condition: $\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \mathbf{x}(0) = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$.

The solution is hence $\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = e^{-t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

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The solution is hence $\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = e^{-t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Notice that $\lim_{t \to +\infty} \mathbf{x}(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

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Let $\mathbf{A} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. Find the solution of the IVP: $\mathbf{x}' = \mathbf{A}\mathbf{x}$ with $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

As above, the general solution is $\mathbf{x}(t) = e^{-t} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$.

Determine the value of C_1, C_2 from the initial condition: $\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \mathbf{x}(0) = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$.

The solution is hence $\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = e^{-t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Notice that $\lim_{t\to+\infty} \mathbf{x}(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Since $\frac{x_2}{x_1} = \frac{e^{-t}2}{e^{-t}} = 2$, the trajectory lies on
the straight-line $x_2 = 2x_1$.

Example: $[\lambda = -1]$

Let $\mathbf{A} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. Find the solution of the IVP: $\mathbf{x}' = \mathbf{A}\mathbf{x}$ with $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

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Since $x_1(t) = e^{-t} > 0$, the trajectory describes the red half-line in the picture, from ∞ towards $(0, 0)$.

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Notice that $\lim_{t \to +\infty} \mathbf{x}(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

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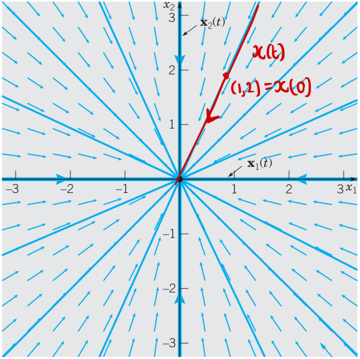
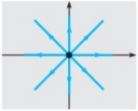
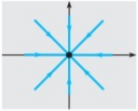
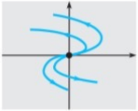
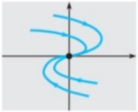
Since $x_1(t) = e^{-t} > 0$, the trajectory describes the red half-line in the picture, from ∞ towards $(0, 0)$.



Summary of case IV

TABLE 3.5.1 Phase portraits for $\mathbf{x}' = \mathbf{A}\mathbf{x}$ when $\mathbf{A}$ has a single repeated eigenvalue.

Nature of **A** and Eigenvalues	Sample Phase Portrait	Type of Critical Point	Stability
$\mathbf{A} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ $\lambda > 0$		(0, 0) is an **unstable proper node.** *Note*: (0, 0) is also called an **unstable star node.**	Unstable
$\mathbf{A} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ $\lambda < 0$		(0, 0) is a **stable proper node.** *Note*: (0, 0) is also called a **stable star node.**	Asymptotically stable
A is not diagonal. $\lambda > 0$		(0, 0) is an **unstable improper node.** *Note*: (0, 0) is also called an **unstable degenerate node.**	Unstable
A is not diagonal. $\lambda < 0$		(0, 0) is a **stable improper node.** *Note*: (0, 0) is also called a **stable degenerate node.**	Asymptotically stable

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The case of non-homogenous systems $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{b}$

$$\mathbf{x}' = \mathbf{A}(\mathbf{x} - \mathbf{v}) \quad \text{where} \quad \begin{cases} \mathbf{x} = \mathbf{x}(t) & \text{is the unknown matrix function} \\ \mathbf{A} & \text{is a constant matrix} \\ \mathbf{v} & \text{is a constant vector} \end{cases}$$

Solution method: *[Example on handout]*

- Set $\mathbf{y}(t) = \mathbf{x}(t) - \mathbf{v}$. Then $\mathbf{y}$ satisfies $\mathbf{y}' = \mathbf{A}\mathbf{y}$.
- Solve $\mathbf{y}' = \mathbf{A}\mathbf{y}$ for the general solution $\mathbf{y}(t) = C_1\mathbf{y}_1(t) + C_1\mathbf{y}_2(t)$ with C_1, C_2 constants.
- Then $\mathbf{x}(t) = \mathbf{y}(t) + \mathbf{v}$ is the solution of the initial system.

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$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{b} \quad \text{where} \quad \begin{cases} \mathbf{x} = \mathbf{x}(t) & \text{is the unknown matrix function} \\ \mathbf{A} & \text{is a constant matrix with } \det \mathbf{A} \neq 0 \\ \mathbf{b} & \text{is a constant vector} \end{cases}$$

Solution method: *[Example on handout]*

- Solve $\mathbf{A}\mathbf{x} + \mathbf{b} = \mathbf{0}$. The solution is $\mathbf{x}_{\text{eq}}$. Hence $\mathbf{A}\mathbf{x}_{\text{eq}} + \mathbf{b} = \mathbf{0}$, i.e. $\mathbf{b} = -\mathbf{A}\mathbf{x}_{\text{eq}}$.
- Substitute in the system, which becomes $\mathbf{x}' = \mathbf{A}(\mathbf{x} - \mathbf{x}_{\text{eq}})$.
- Solve as before.

The case of non-homogenous systems $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{b}$

$$\mathbf{x}' = \mathbf{A}(\mathbf{x} - \mathbf{v}) \quad \text{where} \quad \begin{cases} \mathbf{x} = \mathbf{x}(t) & \text{is the unknown matrix function} \\ \mathbf{A} & \text{is a constant matrix} \\ \mathbf{v} & \text{is a constant vector} \end{cases}$$

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- Substitute in the system, which becomes $\mathbf{x}' = \mathbf{A}(\mathbf{x} - \mathbf{x}_{\text{eq}})$.
- Solve as before.

Conclusion: translation of $\mathbf{x}_{\text{eq}}$ to $(0, 0)$ reduces the analysis (solutions and stability) of $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{b}$ to that of a homogenous system, as before in this section.