EXAMPLE (SECTION 3.6)

$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -xy \\ x^2 \end{pmatrix}$ i.e. $\mathbf{x}' = \mathbf{f}(\mathbf{x}) = \begin{pmatrix} -xy \\ x^2 \end{pmatrix}$

- Find all critical points (or equilibrium solutions)

  The critical points are the constant solutions of the system ($\mathbf{x}' = \mathbf{0}$)

  i.e. $\mathbf{f}(\mathbf{x}) = \mathbf{0}$ i.e. $\begin{cases} -xy = 0 \\ x^2 = 0 \end{cases}$, $\Rightarrow x = 0$, yielding $0=0$ identity in the 1st eq.

  Thus: each constant function: $\begin{cases} x = 0 \\ y = \text{constant} \end{cases}$ is a critical point

  In the phase plane, each point of the y-axis is a critical point.

- In the system of DE's, x' means $\frac{dx}{dt}$, y' means $\frac{dy}{dt}$. We eliminate the t-variable from the system, to get a 1st order DE of the form $\frac{dy}{dx}$ = function of x and y.

  Formally, the equality

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}.$$

is justified by the chain rule, because $x' \frac{dy}{dx} = \frac{dx}{dt}\frac{dy}{dx} = \frac{dy}{dt} = y'$.

Replace $x' = -xy$ and $y' = x^2$ in $x' \frac{dy}{dx} = y'$ and get $-xy \frac{dy}{dx} = x^2$

Since $x=0$ yields the critical points (which we have already determined, we can suppose $x \neq 0$ and divide by x. We get

$$-y \frac{dy}{dx} = x \qquad \text{This is a separable DE.}$$

Integrate both sides wrt x, $\int -y\underbrace{y' dx}_{dy} = \int x dx$, i.e. $-y^2 = x^2 + C$,

i.e. $x^2 + y^2 = c$, c constant.

i.e. $H(x,y) = c$ where $H(x,y) = x^2 + y^2$

Level curves: circles of center $(0,0)$ and radius $\sqrt{c}$, $c \geq 0$

The trajectory through $(2,0)$ is obtained from $c = H(2,0) = 4 \Rightarrow x^2 + y^2 = 4$.

The trajectory through $(2,0)$ cannot cross the y-axis (equilibrium points). Thus it is $x = \sqrt{y^2 - 4}$. The tangent vector to the trajectory at $(2,0)$ is $(x', y')\big|_{(2,0)} = (-xy, x^2)\big|_{(2,0)}$ $= (0,4)$. Thus the orientation of the trajectory is as shown.

each point of the y-axis is a critical point