EXAMPLES (SECTION 4.5)

- $y'' - 3y' - 4y = 4t^2$

The general solution is of the form $y(t) = y_c(t) + Y_1(t)$ where

- $y_c(t)$ is the complementary solution (= general solution of $y'' - 3y' - 4y = 0$)
- $Y_1(t)$ is a particular solution of the given DE.

To find $y_c(t)$: characteristic equation of $y'' - 3y' - 4y = 0$ is $\lambda^2 - 3\lambda - 4 = 0$
$(\lambda - 4)(\lambda + 1)$

Its roots are $\lambda_1 = 4$ and $\lambda_2 = -1$ (real distinct roots)

Thus $y_c(t) = C_1 e^{4t} + C_2 e^{-t}$

To find $Y_1(t)$: the nonhomogeneous term $4t^2$ is a polynomial of degree 2 which is not a solution of the homogenous DE. The table suggests to choose $Y_1(t) = At^2 + Bt + C$ (= polynomial of same degree as t^2).

where A, B, C are parameters which we are going to determine by substituting $Y_1(t)$ in the DE

$Y_1 = At^2 + Bt + C, \quad Y_1' = 2At + B, \quad Y_1'' = 2A$

Substitute into the DE:

$$Y_1'' - 3Y_1' - 4Y_1 = 4t^2 \iff 2A - 3(2At + B) - 4(At^2 + Bt + C) = 4t^2$$

$$\iff -4At^2 - (6A + 4B)t + 2A - 3B - 4C = 4t^2$$

$$\iff \begin{cases} -4A = 4 \\ 3A + 2B = 0 \\ 2A - 3B - 4C = 0 \end{cases} \iff \begin{cases} A = -1 \\ B = 3/2 \\ C = \frac{1}{4}(2A - 3B) = \frac{1}{4}\left(-2 - \frac{9}{2}\right) = -\frac{13}{8} \end{cases}$$

Hence: $Y_1(t) = -t^2 + \frac{3}{2}t + \frac{13}{8}$

Conclusion: the general solution of the DE is

$$y(t) = C_1 e^{4t} + C_2 e^{-t} - t^2 + \frac{3}{2}t - \frac{13}{8},$$ where C_1, C_2 are arbitrary constants.

- $y'' - 3y' - 4y = e^{-t}$

The associated homogeneous DE $y'' - 3y' - 4y = 0$ is equal to that of the previous example. We only need to find a particular solution, say

$y_2(t)$. In this case, the non homogeneous term is e^{-t}, and e^{-t} is a solution of $y''-3y'-4y=0$, but te^{-t} is not.
Hence $y_2(t)=Ate^{-t}$ is the guess according to the table.
Substitute in the DE to find A:

$$y_2(t)=Ate^{-t}$$
$$y_2'(t)=A(e^{-t}-te^{-t})=A(1-t)e^{-t}$$
$$y_2''(t)=-Ae^{-t}-A(1-t)e^{-t}$$
$$=-2Ae^{-t}+Ate^{-t}$$

$$y_2''-3y_2'-4y_2=e^{-t} \iff -2Ae^{-t}+Ate^{-t}-3Ae^{-t}+3Ate^{-t}-4Ate^{-t}=e^{-t}$$
$$\iff -2A+\cancel{At}-3A+\cancel{3At}-\cancel{4At}=1 \quad (\text{because } e^{-t}\neq 0 \ \forall t)$$
$$\iff -5A=1$$
$$\iff A=-\frac{1}{5}$$

Thus: $y_2(t)=-\frac{1}{5}te^{-t}$ and $y(t)=C_1e^{4t}+C_2e^{-t}-\frac{1}{5}te^{-t}$.

- $y''-3y'-4y=4t^2+e^{-t}$

  By the superposition principle, a particular solution of this DE is the sum y_1+y_2 of the particular solutions found above.
  In this case, the general solution is therefore
  $y(t)=C_1e^{4t}+C_2e^{-t}-t^2+\frac{3}{2}t-\frac{13}{8}-\frac{1}{5}te^{-t}$.