EXAMPLES (SECTION 4.7)

Find the general solution of the system of DE's : $\boldsymbol{x}'(t) = \underbrace{\begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}}_{A} \boldsymbol{x}(t) + \underbrace{\begin{pmatrix} t \\ 2t \end{pmatrix}}_{\boldsymbol{g}(t)}$

The general solution is of the form

$$\boldsymbol{x}(t) = C_1 \boldsymbol{x}_1(t) + C_2 \boldsymbol{x}_2(t) + \boldsymbol{x}_p(t)$$

where $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ form a fundamental system of solutions of $\boldsymbol{x}' = A\boldsymbol{x}$ and $\boldsymbol{x}_p$ is a particular solution of the given nonhomogeneous system:

- characteristic equation of A: $\begin{vmatrix} 1-\lambda & -1 \\ 2 & 4-\lambda \end{vmatrix} = 0$ i.e. $\lambda^2 - 5\lambda + 6 = 0$

  Two distinct real eigenvalues $\lambda_1 = 2$, $\lambda_2 = 3$.

  Eigenvectors for $\lambda_1 = 2$: $\begin{pmatrix} -1 & -1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ i.e. $x_1 + x_2 = 0$, i.e. $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = C\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ with $C \neq 0$

  Set $\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Hence $\boldsymbol{x}_1(t) = e^{2t}\boldsymbol{v}_1$.

  Eigenvectors for $\lambda_2 = 3$: $\begin{pmatrix} -2 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, i.e. $2x_1 + x_2 = 0$, i.e. $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = C\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ with $C \neq 0$

  Set $\boldsymbol{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$. Hence $\boldsymbol{x}_2(t) = e^{3t}\boldsymbol{v}_2$.

- $X(t) = \begin{pmatrix} e^{2t} & e^{3t} \\ -e^{2t} & -2e^{3t} \end{pmatrix}$ invertible with $W[\boldsymbol{x}_1, \boldsymbol{x}_2](t) = e^{5t}\begin{vmatrix} 1 & 1 \\ -1 & -2 \end{vmatrix} = -e^{5t}$

  $$X(t)^{-1} = \frac{1}{W[\boldsymbol{x}_1, \boldsymbol{x}_2](t)}\begin{pmatrix} -2e^{3t} & -e^{3t} \\ e^{2t} & e^{2t} \end{pmatrix} = e^{-5t}\begin{pmatrix} 2e^{3t} & e^{3t} \\ -e^{2t} & -e^{2t} \end{pmatrix} = \begin{pmatrix} 2e^{-2t} & e^{-2t} \\ -e^{-3t} & -e^{-3t} \end{pmatrix}$$

  $$X(t)^{-1}\boldsymbol{g}(t) = \begin{pmatrix} 2e^{-2t} & e^{-2t} \\ -e^{-3t} & -e^{-3t} \end{pmatrix}\begin{pmatrix} t \\ 2t \end{pmatrix} = \begin{pmatrix} 4te^{-2t} \\ -3te^{-3t} \end{pmatrix}$$

$$\boldsymbol{x}_p(t) = X(t)\int X^{-1}(t)\boldsymbol{g}(t)\,dt = \begin{pmatrix} e^{2t} & e^{3t} \\ -e^{2t} & -2e^{3t} \end{pmatrix}\int \begin{pmatrix} 4te^{-2t} \\ -3te^{-3t} \end{pmatrix} dt$$

$\left[\int te^{-at}\,dt = -\frac{t}{a}e^{-at} + \frac{1}{a}\int e^{-at}\,dt = -\frac{t}{a}e^{-at} - \frac{1}{a^2}e^{-at} + C = \frac{-e^{-at}}{a}\left(t + \frac{1}{a}\right) + C \right]$ $(a \neq 0)$

$$= \begin{pmatrix} e^{2t} & e^{3t} \\ -e^{2t} & -2e^{3t} \end{pmatrix}\begin{pmatrix} -2e^{-2t}\left(t + \frac{1}{2}\right) \\ e^{-3t}\left(t + \frac{1}{3}\right) \end{pmatrix} = \begin{pmatrix} -2\left(t+\frac{1}{2}\right) + \left(t + \frac{1}{3}\right) \\ 2\left(t+\frac{1}{2}\right) - 2\left(t+\frac{1}{3}\right) \end{pmatrix} = \begin{pmatrix} -t - 2/3 \\ 1/3 \end{pmatrix}$$

Conclusion: general solution is

$$\boldsymbol{x}(t) = C_1 e^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2 e^{3t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} + \begin{pmatrix} -t - 2/3 \\ 1/3 \end{pmatrix}$$

EXAMPLE:

Find a particular solution of $y'' - 3y' - 4y = \underbrace{e^{-t}}_{g(t)}$ using the method of variations of parameters.

The functions $y_1(t) = e^{4t}$ and $y_2(t) = e^{-t}$ form a fundamental system of solutions of $y'' - 3y' - 4y = 0$ (on $I = \mathbb{R}$); see section 4.5. Their Wronskian is

$$W[y_1, y_2](t) = \begin{vmatrix} e^{4t} & e^{-t} \\ 4e^{4t} & -e^{-t} \end{vmatrix} = -e^{3t} - 4e^{3t} = -5e^{3t}.$$

By the method of variation of parameters, a particular solution of $y'' - 3y' - 4y = e^{-t}$ is

$$Y(t) = -y_1(t) \int \frac{y_2(t)\, g(t)}{W[y_1, y_2](t)}\, dt + y_2(t) \int \frac{y_1(t)\, g(t)}{W[y_1, y_2](t)}\, dt$$

$$= -e^{4t} \int \frac{e^{-2t}}{(-5e^{3t})}\, dt + e^{-t} \int \frac{e^{3t}}{(-5e^{3t})}\, dt =$$

$$= \frac{1}{5} e^{4t} \int e^{-5t} dt - \frac{1}{5} t e^{-t} = \frac{1}{5} e^{4t}\left(-\frac{1}{5} e^{-5t}\right) - \frac{1}{5} t e^{-t} = -\frac{1}{25} e^{-t} - \frac{1}{5} t e^{-t}$$

REMARK: we can compare this solution with the solution $Y_1(t) = -\frac{1}{5} t e^{-t}$ we found, for the same DE, in section 4.5, using the method of undetermined constants. Notice that $Y(t) = -\frac{1}{25} e^{-t} + Y_1(t)$, and hence $Y(t) - Y_1(t) = -\frac{1}{25} e^{-t}$. This is a solution of the homogeneous equation $y'' - 3y' - 4y = 0$ (it is a multiple of $y_2(t) = e^{-t}$). This agrees with what we pointed out in Theorem 4.5.2 - see pages 2-3 of the slides of section 4.5.