Shifted systems of first-order linear DE's with constant coefficients

$\mathbf{x}' = \mathbf{A}(\mathbf{x} - \mathbf{v})$	where	$\mathbf{x} = \mathbf{x}(t)$	is the unknown matrix function
		$\mathbf{A}$	is a constant matrix
		$\mathbf{v}$	is a constant vector

Solution method:

- Set $\mathbf{y}(t) = \mathbf{x}(t) - \mathbf{v}$. Then $\mathbf{y}$ satisfies $\mathbf{y}' = \mathbf{A}\mathbf{y}$.
- Solve $\mathbf{y}' = \mathbf{A}\mathbf{y}$ for the general solution $\mathbf{y}(t) = C_1\mathbf{y}_1(t) + C_1\mathbf{y}_2(t)$ with C_1, C_2 constants.
- Then $\mathbf{x}(t) = \mathbf{y}(t) + \mathbf{v}$ is the solution of the initial system.

Example: Solve $\mathbf{x}' = \mathbf{A}(\mathbf{x} - \mathbf{v})$ where $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$.

Solution:

- Set $\mathbf{y} = \mathbf{x} - \mathbf{v}$.
- Solve $\mathbf{y}' = \mathbf{A}\mathbf{y}$.
  The characteristic polynomial of $\mathbf{A}$ is
  $$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 3 = (\lambda - 1)(\lambda - 3)\,.$$
  The eigenvalues of $\mathbf{A}$ are $\lambda_1 = 1$ and $\lambda_2 = 3$.
  $\mathbf{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ for the eigenvalue $\lambda_1 = 1$.
  $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ for the eigenvalue $\lambda_2 = 3$.
  The general solution is $\mathbf{y}(t) = C_1 e^t \begin{pmatrix} -1 \\ 1 \end{pmatrix} + C_2 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ where C_1, C_2 are constants.
- The general solution of the initial system of DE's is
  $$\mathbf{x}(t) = C_1 e^t \begin{pmatrix} -1 \\ 1 \end{pmatrix} + C_2 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$ where C_1, C_2 are constants.

$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{b}$	where	$\mathbf{x} = \mathbf{x}(t)$	is the unknown matrix function
		$\mathbf{A}$	is a constant matrix with $\det \mathbf{A} \neq 0$
		$\mathbf{b}$	is a constant vector

Solution method:

- Solve $\mathbf{A}\mathbf{x} + \mathbf{b} = \mathbf{0}$. Let $\mathbf{v}$ denote the solution. Hence $\mathbf{A}\mathbf{v} + \mathbf{b} = \mathbf{0}$, i.e. $\mathbf{b} = -\mathbf{A}\mathbf{v}$.
- Substitute in the system, which becomes $\mathbf{x}' = \mathbf{A}(\mathbf{x} - \mathbf{v})$.
- Solve as before.

Example: Solve $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{b}$ where $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$.

Solution:

- Solve $\mathbf{A}\mathbf{x} + \mathbf{b} = \mathbf{0}$, i.e. $\mathbf{A}\mathbf{x} = -\mathbf{b}$. This is the system $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, with solution $x_1 = -\frac{1}{3}$, $x_2 = \frac{2}{3}$. So $\mathbf{v} = \begin{pmatrix} -1/3 \\ 2/3 \end{pmatrix}$.
- Set $\mathbf{y} = \mathbf{x} - \mathbf{v}$.
- Solve $\mathbf{y}' = \mathbf{A}\mathbf{y}$. As in the previous example, The general solution is $\mathbf{y}(t) = C_1 e^t \begin{pmatrix} -1 \\ 1 \end{pmatrix} + C_2 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ where C_1, C_2 are constants.
- The general solution of $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{b}$ is hence
  $$\begin{aligned} \mathbf{x}(t) &= C_1 e^t \begin{pmatrix} -1 \\ 1 \end{pmatrix} + C_2 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \mathbf{v} \\ &= C_1 e^t \begin{pmatrix} -1 \\ 1 \end{pmatrix} + C_2 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -1/3 \\ 2/3 \end{pmatrix}, \end{aligned}$$
  where C_1, C_2 are constants.