GEORGIA TECH LORRAINE - METZ
MATH 2552
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DIFFERENTIAL EQUATIONS
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MIDTERM II

____________________________________ (Name and Signature)

1.

a. (2 points) Show that the function $f(t) = e^{3t}$ is of exponential order.

$$\lim_{t\to+\infty} \frac{f(t)}{e^{4t}} = \lim_{t\to+\infty} \frac{e^{3t}}{e^{4t}} = \lim_{t\to+\infty} e^{-t} = 0$$

So $f(t)$ is of exponential order.

b. (2 points) Show that the function $f(t) = e^{t^3}$ is not of exponential order.

$$\lim_{t\to+\infty} \frac{f(t)}{e^{at}} = \lim_{t\to+\infty} \frac{e^{t^3}}{e^{at}} = \lim_{t\to+\infty} e^{t^3-at} = +\infty \text{ for any real number } a$$

So $f(t)$ is not of exponential order.

2. (4 points) Using the definition, find the Laplace transform of $f(t) = \begin{cases} e^{5t} & \text{if } 0 \le t \le 2 \\ 1 & \text{if } t > 2 \end{cases}$

$$\mathcal{L}\{f\}(s) = \int_0^{+\infty} e^{-ts} f(t)\,dt = \int_0^{2} e^{-ts} f(t)\,dt + \int_2^{+\infty} e^{-ts} f(t)\,dt$$

$$= \int_0^{2} e^{-ts} e^{5t}\,dt + \int_2^{+\infty} e^{-ts}\,dt$$

$$= \int_0^{2} e^{-(s-5)t}\,dt + \lim_{A\to+\infty} \int_2^{A} e^{-ts}\,dt$$

$$= \left[\frac{1}{5-s} e^{(5-s)t}\right]_0^2 + \lim_{A\to+\infty} \left[-\frac{1}{s} e^{-st}\right]_2^A$$

$$\boxed{\mathcal{L}\{f\}(s) = \frac{e^{2(5-s)}}{5-s} - \frac{1}{5-s} + \frac{1}{s} e^{-2s} \quad \text{for } s > 0}$$

3. a. (4 points) Find real numbers a, b, c and d such that $\dfrac{s^3+2s+2}{s(s^2+1)(s+2)} = \dfrac{a}{s} + \dfrac{bs+c}{s^2+1} + \dfrac{d}{s+2}$.

We could proceed by identification, but we could also proceed as follows.

Denote $Y(s) = \dfrac{s^2+2s+2}{s(s^2+1)(s+2)}$.

Evaluate $sY(s)$ at $s=0 \iff \dfrac{0^3+2\times 0+2}{(0^2+1)(0+2)} = a \iff a=1$

$(s+2)Y(s)$ at $s=-2 \iff \dfrac{(-2)^3+2\times(-2)+2}{(-2)\times((-2)^2+1)} = d \iff d=1$

$(s-i)Y(s)$ at $s=i \iff \dfrac{i^3+2\times i+2}{i(i+i)(i+2)} = \dfrac{bi+c}{i+i} \iff b=-1,\ c=0$

b. (4 points) Find the inverse Laplace transform of $Y(s) = \dfrac{s^3+2s+2}{s(s^2+1)(s+2)}$.

From the previous question, we have

$$Y(s) = \frac{1}{s} + \frac{1}{s+2} - \frac{s}{s^2+1} = \frac{1}{s} + \frac{1}{s-(-2)} - \frac{s}{s^2+1^2}$$

From the table of inverse Laplace transform we get that:

$$\mathcal{L}^{-1}(Y)(t) = \mathcal{L}^{-1}\left(\frac{1}{s}\right)(t) + \mathcal{L}^{-1}\left(\frac{1}{s-(-2)}\right)(t) - \mathcal{L}^{-1}\left\{\frac{s}{s^2+1^2}\right\}(t)$$

$$= 1 + e^{-2t} - \cos(t)$$

4. Consider the second order differential equation $y'' + y' - 2y = e^{2t}$.

a. (3 points) Solve the homogeneous differential equation $y'' + y' - 2y = 0$.

characteristic equation: $\lambda^2 + \lambda - 2 = 0$

solutions are $\lambda_1 = 1$ and $\lambda_2 = -2$.

the general solution of the homogeneous equation is of the form: $\boxed{c_1 e^{t} + c_2 e^{-2t}}$ where c_1, c_2 are constant numbers

b. (4 points) Find a particular solution of the full differential equation $y'' + y' - 2y = e^{2t}$.

Hint: recall the formula $Y(t) = -y_1(t)\int \frac{y_2(t)g(t)}{W[y_1,y_2](t)}dt + y_2(t)\int \frac{y_1(t)g(t)}{W[y_1,y_2](t)}dt$

Set $y_1(t) = e^{t}$ and $y_2(t) = e^{-2t}$.

Wronskian $W[y_1,y_2](t) = \begin{vmatrix} e^{t} & e^{-2t} \\ e^{t} & -2e^{-2t} \end{vmatrix} = -2e^{t} \times e^{-2t} - e^{t}e^{-2t} = -3e^{-t}$

$W[y_1,y_2](t) \neq 0$ for any real number t. So $\{y_1, y_2\}$ is a set of fundamental solutions of the homogeneous DE above.

Using the hint we know that a particular solution to the full DE is given by:

$$Y(t) = -e^{t}\int \frac{e^{-2t} \times e^{2t}}{-3e^{-t}}dt + e^{-2t}\int \frac{e^{t} \times e^{2t}}{-3e^{-t}}dt$$

$$= \frac{1}{3}e^{t}\int e^{t}dt - \frac{1}{3}e^{-2t}\int e^{4t}dt$$

$$= \frac{1}{3}e^{t} \times e^{t} - \frac{1}{3}e^{-2t} \times \frac{1}{4}e^{4t} = \frac{1}{3}e^{2t} - \frac{1}{12}e^{2t}$$

$$\boxed{Y(t) = \frac{1}{4}e^{2t}}$$

c. (2 points) Deduce the general solution of the full differential equation $y'' + y' - 2y = e^{2t}$.

The general solution to the full differential equation is therefore:

$$\boxed{c_1 e^{t} + c_2 e^{-2t} + \frac{1}{4}e^{2t}}$$