Georgia Tech – Lorraine
Spring 2020
Differential Equations
Math 2552
2/12/2020

Last Name:
First Name:

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Midterm n^0 1 (50 minutes)

- *Write your answers in the provided answer box.*
- *Show your work and justify your answers.*
- *Calculators, notes, cell phones, books are not allowed.*
- *Please do not use red or pink ink.*

Maximum: 25 points

Exercise 1 [**1+1+1+1+5 points**]
Consider the differential equation: $y' = x^2y^2 - 4x^2$.

(a) Is this differential equation separable? ANSWER: Yes ☑ No ☐.

Justify:

$y' = x^2(y^2-4)$ is of the form $y' = f(x)g(y)$ where $f(x) = x^2$ and $g(y) = y^2-4$

(b) Is this differential equation linear? ANSWER: Yes ☐ No ☑.

Justify. If it is linear, write it in standard form.

It is not linear because of the term y^2

(c) Is this differential equation exact? ANSWER: Yes ☑ No ☐.

Justify:

Every separable differential equation is exact. To check this for the test for exactness; rewrite the DE as $-x^2 + \frac{1}{y^2-4} y' = 0$ and set $M(x,y) = -x^2$ and $N(x,y) = \frac{1}{y^2-4}$. Then $\frac{\partial M}{\partial y} = 0 = \frac{\partial N}{\partial x}$

(d) Is this differential equation a Bernoulli differential equation? ANSWER: Yes ☐ No ☑.

Justify:

A Bernoulli DE is of the form $y' + q(x)y = r(x)y^n$ for some real number n. Our DE cannot be brought to this form because of the term $-4x^2$ (not multiplying y', y or y^2)

Please turn: Question (e) on the following page ⟶

Exercise 1 (continued)

(e) Solve the differential equation $y' = x^2y^2 - 4x^2$.

[*Hint:* you might want to use a partial fraction decomposition to compute one of the integrals]

$y' = x^2(y^2-4)$ separable 1st order DE. Notice that $y^2-4=(y-2)(y+2)$.
$y=\pm 2$ are constant solutions. If $y \neq \pm 2$, i.e. $y^2-4\neq 0$, can divide both sides by y^2-4 and get: $\frac{1}{(y-2)(y+2)} y' = x^2$. Integrate both sides wrt x:

$$\int \frac{1}{(y-2)(y+2)} \underbrace{\frac{dy}{dx} dx}_{dy} = \int x^2 dx, \text{ i.e. } \int \frac{1}{(y-2)(y+2)} dy = \int x^2 dx \quad (*)$$

Partial fraction decomposition: $\frac{1}{(y-2)(y+2)} = \frac{A}{y-2} + \frac{B}{y+2} = \frac{(A+B)y+2(A-B)}{(y-2)(y+2)}$

yields: $(A+B)y + 2(A-B) = 1$. This must hold for all y, which is possible only if

$$\begin{cases} A+B=0 \\ 2(A-B)=1 \end{cases} \text{ i.e. } A=\frac{1}{4}, B=-\frac{1}{4}$$

Thus $\int \frac{1}{(y-2)(y+2)} dy = \frac{1}{4}\int \frac{dy}{y-2} - \frac{1}{4}\int\frac{dy}{y+2} = \frac{1}{4}\ln|y-2| - \frac{1}{4}\ln|y+2| + \text{const.}$

$= \frac{1}{4}\ln\left|\frac{y-2}{y+2}\right| + \text{const.}$

and $(*)$ is solved by

$\frac{1}{4}\ln\left|\frac{y-2}{y+2}\right| = \frac{1}{3}x^2 + C_0$, C_0 constant

i.e. $\ln\left|\frac{y-2}{y+2}\right| = \frac{4}{3}x^2 + C_1$, i.e. $\left|\frac{y-2}{y+2}\right| = e^{C_1}e^{\frac{4}{3}x^2}$, i.e. $\frac{y-2}{y+2} = C\,e^{\frac{4}{3}x^2}$

where C is a constant $\neq 0$.

To solve for y, we multiply both sides by $y+2$ and obtain

$y-2 = C(y+2)e^{4/3x^2}$. Then collect y and get $y(1-Ce^{4/3x^2}) = 2(1+Ce^{4/3x^2})$.

Thus $y = 2\frac{1+Ce^{\frac{4}{3}x^2}}{1-Ce^{\frac{4}{3}x^2}}$, where C is an arbitrary constant ($C=0$ corresponds to the constant solution $y=2$)

Answer:

$y = 2\frac{1+Ce^{4/3x^2}}{1-Ce^{4/3x^2}}$, C a constant; additional constant solution $y=-2$

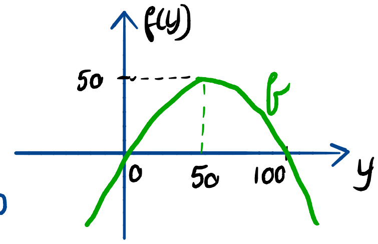
Exercise 2 [**3 points**] The size of a population of wolves is modeled by the differential equation

$$\frac{dy}{dt} = -\frac{1}{50}y(y-100)$$

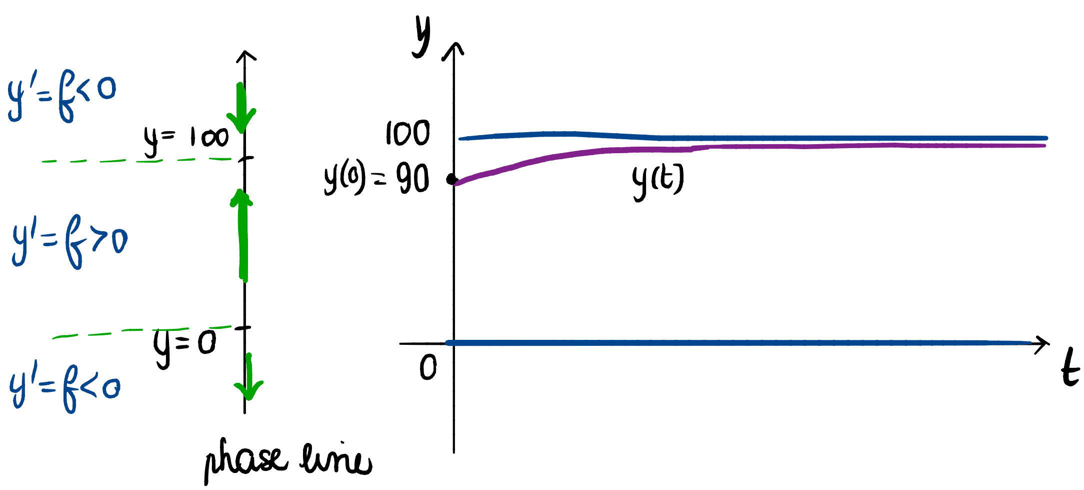
where $y = y(t)$ is the size of the population at time t, ~~and k is a positive constant~~.
Estimate the size of the population after a long period of time if the initial size is 90 wolves.
Justify your answer.

The DE modeling the size $y(t)$ of the population at time t is an autonomous 1st order DE of the form $\frac{dy}{dt} = f(y)$ where $f(y) = -\frac{1}{50}y(y-100)$

The graph of f is a parabola concave down and the roots of $f(y)=0$ are $y=0$, $y=100$. The vertex of the parabola is (x_V, y_V) where $x_V = 50 = \frac{100}{2}$ and $y_V = f(50) = 50$. A sketch of the graph of f



Equilibrium solutions of the DE are the solutions of $f(y)=0$, i.e. $y=0$, $y=100$



— = qualitative graph of the solution of the IVP: $\frac{dy}{dt} = f(y)$ with initial condition $y(0)=90$

The size of the population after a long period of time tends to stabilize at the value of 100 units (without depassing this value)

Answer: The size of the population after a long period of time is 100 wolves.

Exercise 3 **[1+2+5+2+(1 bonus) points]**

Consider the matrix $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 3 & 2 \end{pmatrix}$.

(a) Compute the trace and the determinant of $\mathbf{A}$.

$$\operatorname{trace}\begin{pmatrix} 0 & 1 \\ 3 & 2 \end{pmatrix} = 0+2=2 \; ; \; \det A = \begin{vmatrix} 0 & 1 \\ 3 & 2 \end{vmatrix} = 0\cdot 2 - 1\cdot 3 = -3$$

ANSWERS: The trace of $\mathbf{A}$ is 2 The determinant of $\mathbf{A}$ is -3

(b) Compute the characteristic polynomial of $\mathbf{A}$ and determine the eigenvalues of $\mathbf{A}$.

$$\det(A-\lambda I) = \begin{vmatrix} -\lambda & 1 \\ 3 & 2-\lambda \end{vmatrix} = -\lambda(2-\lambda)-3 = \lambda^2-2\lambda-3 = (\lambda-3)(\lambda+1)$$

The eigenvalues of A are the roots of the characteristic equation $\det(A-\lambda I)=0$

i.e. $(\lambda-3)(\lambda+1)=0$, yielding $\lambda_1=3$, $\lambda_2=-1$

ANSWERS: Characteristic polynomial: $\lambda^2-2\lambda-3$, i.e. $(\lambda-3)(\lambda+1)$ Eigenvalues: $\lambda_1=3$, $\lambda_2=-1$

(c) Find the general solution of the system of differential equations $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

Eigenvectors for eigenvalue $\lambda_1=3$: $(A-\lambda_1 I)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, i.e. $\begin{pmatrix} -3 & 1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, i.e. $-3x_1+x_2=0$

So $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ 3x_1 \end{pmatrix} = x_1\begin{pmatrix} 1 \\ 3 \end{pmatrix}$, $x_1 \in \mathbb{R}$

Fixing the eigenvector $\underline{v}_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$, we obtain the solution $\underline{x}_1(t) = e^{3t}\begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

Eigenvectors for the eigenvalue $\lambda_2=-1$: $(A-\lambda_2 I)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, i.e. $\begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, i.e. $x_1+x_2=0$

So $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ -x_1 \end{pmatrix} = x_1\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $x_1 \in \mathbb{R}$.

Fixing the eigenvector $\underline{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, we obtain the solution $\underline{x}_2(t) = e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Since $\lambda_1=3$ and $\lambda_2=-1$ are real and distinct, $\underline{x}_1$ and $\underline{x}_2$ form a fundamental set of solutions of the system $\underline{x}'=A\underline{x}$. The general solution is therefore

ANSWER: $\underline{x}(t) = C_1\underline{x}_1(t)+C_2\underline{x}_2(t) = C_1e^{3t}\begin{pmatrix} 1 \\ 3 \end{pmatrix}+C_2e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, C_1, C_2 constants

(d) Find the solution of the initial value problem $\mathbf{x}' = \mathbf{A}\mathbf{x}$ with the initial condition $\mathbf{x}(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

One can observe that $\underline{x}_2(t)$ satisfies $\underline{x}_2(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. By the uniqueness of the solution of an IVP, this is the solution we are looking for. OR: one can apply the usual method of determining C_1, C_2 so that $\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \underline{x}(0) = C_1\begin{pmatrix} 1 \\ 3 \end{pmatrix}+C_2\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, i.e. $\begin{cases} C_1+C_2=1 \\ 3C_1-C_2=-1 \end{cases}$ yielding $C_1=0$, $C_2=1$

ANSWER: The solution of this IVP is $\underline{x}_2(t) = e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

(e) **Bonus question: 1 additional point**
The system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is equivalent to a second order differential equation $ay'' + by' + cy = 0$. Determine a, b and c.

The system $\underline{x}' = A\underline{x}$, i.e. $\begin{cases} x_1' = x_2 \\ x_2' = 3x_1 + 2x_2 \end{cases}$ can be transformed into an equivalent 2nd order DE with constant coefficients by setting $\begin{cases} x_1 = y \\ x_2 = y' \end{cases}$
The second equation of the system then leads to
$y'' = 3y + 2y'$, i.e. $y'' - 2y' - 3y = 0$.

Thus $(a,b,c) = (1,-2,-3)$ or any non zero multiple of it, since a,b,c are defined only up to multiplication by a non zero constant.

ANSWER:	$(a,b,c) = (1,-2,-3)$ [or any non zero multiple of this vector]

Exercise 4 [**3 points**]
Determine the largest interval I where the solution of the following initial value problem exists and is unique:

$$(e^t - 1)\frac{dy}{dt} + y = \frac{t}{t-2} \qquad \text{with the initial condition} \qquad y(1) = 2$$

Justify your work. *(Do not attempt to solve the differential equation)*

This is a linear DE. Its standard form is $\frac{dy}{dt} + \underbrace{\frac{1}{e^t-1}}_{p(t)} y = \underbrace{\frac{t}{(e^t-1)(t-2)}}_{g(t)}$
The function $p(t)$ is defined and continuous provided $e^t - 1 \neq 0$, i.e. for all $t \neq 0$, i.e. on $(-\infty, 0) \cup (0, +\infty)$
The function $g(t) = \frac{t}{(e^t-1)(t-2)}$ is defined and continuous provided $t \neq 0, 2$
Thus both $p(t)$ and $g(t)$ are def and continuous on $(-\infty; 0) \cup (0; 2) \cup (2; +\infty)$
Thus $I = (0; 2)$, because this is the largest interval entirely contained in $(-\infty; 0) \cup (0; 2) \cup (2; +\infty)$ and containing $t = 1$

ANSWER:	$I = (0; 2)$