Georgia Tech – Lorraine
Spring 2020
Differential Equations
Math 2552
2/12/2020

Last Name:
First Name:

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Midterm n^0 2 (2 hours)

- Please email your solution to *angela.pasquale@univ-lorraine.fr* or *angela.pasquale@georgiatech-metz.fr* today, by 12:30 pm (Atlanta time). Write *"Midterm 2"* in the subject.
- You can write your solution on the same pdf I sent you, for instance if you have a tablet or you can print it. If you need extra space, add as many pages as needed.
  Otherwise, please write your solutions on blank paper. You do not need to copy the questions, just clearly *mark the number of the exercises and their questions* and *separate the different exercises with a horizontal line.*
- Please call your file "yourname-Midterm2".
- Show your work and justify your answers. Please organize your work clearly, neatly, and legibly. Identify your answers.
- You have to solve the problems by yourself, you are not allowed to discuss problems and solutions with other people in any form. Please abide to the Honor Code.
- I will be online during the whole exam time. You can send me messages by email. I will do my best to answer as soon as I can. Sometimes I will be answering to other people. So please be patient. Also, please understand that there are questions to which I cannot answer: for instance, if your solution is correct or not.
- Maximum: 25 points

Exercise 1 [3+1+1+5+1 points]

Consider the linear differential equation $t^2y'' - t(t+2)y' + (t+2)y = 0$ where $t > 0$.

(a) Verify that $y_1(t) = t$ and $y_2(t) = te^t$ are solutions and that they are linearly independent when $t > 0$.

• $y_1(t)=t$, $y_1'(t)=1$, $y_1''(t)=0$: $t^2y_1'' - t(t+2)y_1' + (t+2)y_1 = -t(t+2) + (t+2)t = 0$

Hence $y_1(t)$ is a solution.

• $y_2(t)=te^t$, $y_2'(t) = e^t+te^t$, $y_2''(t) = e^t + e^t + te^t = 2e^t + te^t$

$$t^2y_2'' - t(t+2)y_2' + (t+2)y_2 = t^2(2e^t+te^t) - t(t+2)(e^t+te^t) + (t+2)te^t$$
$$= 2t^2e^t + t^3e^t - t(te^t + 2te^t + 2e^t + t^2e^t) + t^2e^t + 2te^t$$
$$= \cancel{2t^2e^t} + \cancel{t^3e^t} - \cancel{t^2e^t} - \cancel{2t^2e^t} - \cancel{2te^t} - \cancel{t^3e^t} + \cancel{t^2e^t} + \cancel{2te^t} = 0$$

Hence $y_2(t)$ is a solution.

• $W[y_1,y_2](t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} t & te^t \\ 1 & e^t+te^t \end{vmatrix} = te^t + t^2e^t - te^t = t^2e^t \neq 0$ for $t>0$.

Thus y_1, y_2 are lin. indep. for $t>0$

Please turn: Questions (b) to (e) on the following page ⟶

Exercise 1 (continued)

(b) Write the general solution of $t^2y'' - t(t+2)y' + (t+2)y = 0$ for $t > 0$.

$\boxed{y(t) = C_1 y_1(t) + C_2 y_2(t) = C_1 t + C_2 te^t, \ C_1, C_2 \text{ arbitrary constants}}$ (because y_1, y_2 two lin. indep. solutions)

We now consider the non-homogenous differential equation

$$t^2y'' - t(t+2)y' + (t+2)y = t^4e^t\,. \tag{1}$$

(c) Write it in standard form.

$$\boxed{\text{For } t \neq 0: \ y'' - \frac{t+2}{t}y' + \frac{t+2}{t^2}y = t^2e^t}$$

(d) Find a particular solution.

Apply the method of variation of parameters. A particular solution is

$$Y(t) = -y_1(t)\int \frac{y_2(t)g(t)}{W[y_1,y_2](t)}\,dt + y_2(t)\int \frac{y_1(t)g(t)}{W[y_1,y_2](t)}\,dt$$

where $g(t) = t^2e^t = W[y_1,y_2](t)$

$$Y(t) = -t\int te^t\,dt + te^t\int t\,dt$$

$$= -t(te^t - e^t + C_1) + te^t\left(\tfrac{1}{2}t^2 + C_2\right)$$

Integration by parts gives:

$$\int \underset{f}{t}\,\underset{g'}{e^t}\,dt = \underset{f}{t}\,\underset{g}{e^t} - \int \underset{f'}{1}\cdot\underset{g}{e^t}\,dt = te^t - e^t + C_1$$

where C_1 and C_2 are constants which we can fix to be equal to 0

So a particular solution is

$$\boxed{Y(t) = te^t - t^2e^t + \frac{1}{2}t^3e^t}$$

Since te^t is a solution of the associated homogeneous equation,

$$\boxed{Y_0(t) = -t^2e^t + \frac{1}{2}t^3e^t}$$

is another particular solution.

(e) Determine the general solution of (1).

The general solution of (1) is the sum of the general solution of the associated homogeneous equation and a particular solution of (1). Hence it is

$$\boxed{y(t) = C_1 t + C_2 te^t - t^2e^t + \frac{1}{2}t^3e^t, \ C_1, C_2 \text{ constants}, \ t > 0}$$

Please turn ⟶

Exercise 2 [3+1+1+1 points]

Consider the system of linear DE's $\mathbf{x}' = \mathbf{A}\mathbf{x}$ where $\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ and $\mathbf{A} = \begin{pmatrix} 5 & -3 \\ 3 & -5 \end{pmatrix}$.

(a) Find the general solution.

Characteristic equation of A: $\det(A-\lambda I)=0$, i.e. $\begin{vmatrix} 5-\lambda & -3 \\ 3 & -5-\lambda \end{vmatrix} = 0$, i.e. $\underbrace{(\lambda+5)(\lambda-5)}_{\lambda^2-25}+9=0$

i.e. $\lambda^2-16=0$. The eigenvalues of A are hence $\lambda_1 = 4$, $\lambda_2 = -4$.

Eigenvectors for $\lambda_1 = 4$: $(A-4I)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 & -3 \\ 3 & -9 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow v_1 - 3v_2 = 0 \Leftrightarrow \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = v_2\begin{pmatrix} 3 \\ 1 \end{pmatrix}$

Fix $\mathbf{v}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

Eigenvectors for $\lambda_2 = -4$: $(A+4I)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 9 & -3 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow 3v_1 - v_2 = 0 \Leftrightarrow \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = v_1\begin{pmatrix} 1 \\ 3 \end{pmatrix}$

Fix $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

The matrix A is non-defective because it has two distinct real eigenvalues.
The general solution of $\mathbf{x}' = A\mathbf{x}$ is therefore

$$\boxed{\mathbf{x}(t) = C_1 e^{4t}\begin{pmatrix} 3 \\ 1 \end{pmatrix} + C_2 e^{-4t}\begin{pmatrix} 1 \\ 3 \end{pmatrix}, \quad C_1, C_2 \text{ constants}, \ t \in \mathbb{R}}$$

(b) Determine the equilibrium point, identify its type and determine its stability.

$\mathbf{x}' = A\mathbf{x}$ is a homogeneous system of 1st order linear DE's with $\det(A) = \begin{vmatrix} 5 & -3 \\ 3 & -5 \end{vmatrix} = -25+9 = -16 \neq 0$. The system $A\mathbf{x} = \mathbf{0}$ has therefore unique solution $\mathbf{x} = \mathbf{0}$. Thus $(0,0)$ is the unique equilibrium solution. A has two real eigenvalues of opposite signs. The equilibrium solution is then a saddle and it is unstable.

Please turn: Questions (c) to (d) on the following page ⟶

Exercise 2 (continued)

(c) Pick one of the two eigenvalues of $\mathbf{A}$ you determined in (a) and call it λ. Find an eigenvector $\mathbf{v}$ of $\mathbf{A}$ for the eigenvalue λ so that the solution $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$ has first component equal to 1 at $t = 0$.

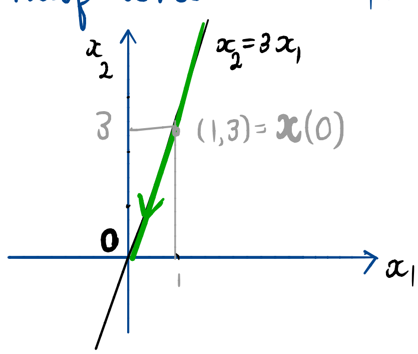
Depending on the choice of λ, two answers are possible:

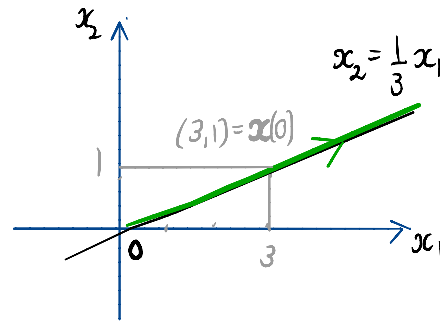
- If we choose $\lambda_2 = -4$ and $\mathbf{v}_2 = \begin{pmatrix}1\\3\end{pmatrix}$, then for any constant $C \neq 0$, the vector $C\mathbf{v}_2$ is an eigenvector of A for the eigenvalue $\lambda_2 = -4$. Set $\mathbf{x}(t) = \begin{pmatrix}x_1(t)\\x_2(t)\end{pmatrix}$. We have to determine the value of C so that for $\mathbf{x}(t) = Ce^{-4t}\begin{pmatrix}1\\3\end{pmatrix}$ we have $x_1(0) = 1$, i.e. $Ce^0 = 1$. Thus $C = 1$ and $\mathbf{v} = \mathbf{v}_2 = \begin{pmatrix}1\\3\end{pmatrix}$
- If we choose $\lambda_1 = 4$ and $\mathbf{v}_1 = \begin{pmatrix}3\\1\end{pmatrix}$, then for any $C \neq 0$ the vector $C\mathbf{v}_1$ is an eigenvector of A. The given condition is that for $\mathbf{x}(t) = Ce^{4t}\begin{pmatrix}3\\1\end{pmatrix}$ we have $x_1(0) = 1$. Thus $1 = 3C$, yielding $C = \frac{1}{3}$ and $\mathbf{v} = \frac{1}{3}\mathbf{v}_1 = \begin{pmatrix}1\\1/3\end{pmatrix}$.

(d) Let $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$ the solution you determined in (c). Sketch its trajectory in the phase plane. *(Do not forget to indicate by an arrowhead the direction of motion along the trajectory).*

Depending on your choice of λ in exercise 2(c), there are two possible answers:

- If $\mathbf{x}(t) = e^{-4t}\begin{pmatrix}1\\3\end{pmatrix}$: since $\frac{x_2(t)}{x_1(t)} = \frac{3e^{-4t}}{e^{-4t}} = 3$, the trajectory of this solution lies on the line $x_2 = 3x_1$. Since $\{x_1(t) = e^{-4t};\, t \in \mathbb{R}\} = (0, +\infty)$ and $\lim_{t\to+\infty} x_1(t) = 0$, the trajectory is the half-line with $x_1 > 0$ [in 1st quadrant] on $x_2 = 3x_1$ oriented towards $(0,0)$:





- If $\mathbf{x}(t) = e^{4t}\begin{pmatrix}1\\1/3\end{pmatrix}$, then $\frac{x_2}{x_1} = \frac{1}{3}$ and the trajectory would be on $x_2 = \frac{1}{3}x_1$; the half-line in $x_1 > 0$, oriented away from $(0,0)$

Please turn ⟶

Exercise 3 [1+3+3+1 points]

A mass of 2 kg is hung from a spring of spring constant $k = 1.85$ N/m. Suppose that it is also attached to a viscous damper that exerts a force of 0.03 N when the velocity of the mass is 0.05 m/s. The mass is pulled down 0.1 m below its equilibrium position and then released. Suppose that there is no external force.

(a) Determine the damping coefficient γ.

$$\gamma = \frac{0{,}03}{0{,}05} \frac{\text{N}}{\text{m/s}} = \frac{3}{5} \frac{\text{Ns}}{\text{m}}$$

(b) Write down the appropriate initial value problem that governs the motion of the mass.

The motion of a mass in a damped unforced spring-mass system is described by the DE: $my'' + \gamma y' + ky = 0$

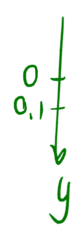
where $y = y(t)$ is the position of the mass along a vertical y-axis, with the positive direction downward and with the origin at the equilibrium position of the mass.

Thus the motion satisfies the IVP

$$2y'' + \frac{3}{5}y' + 1.85y = 0 \quad , \quad y(0) = 0.1 \quad , \quad y'(0) = 0$$

below the equilibrium position (= positive value)

released = no initial velocity



Please turn: Questions (c) to (d) on the following page $\longrightarrow$

Exercise 3 (continued)

(c) Solve the initial value problem and find the position of the mass at any time t.

$2y'' + \frac{3}{5}y' + 1.85y = 0 \quad , y(0) = 0.1 \quad , y'(0) = 0$

$10y'' + 3y' + 9.25y = 0$

Characteristic equation: $10\lambda^2 + 3\lambda + 9.25 = 0$,

Its roots are $\lambda = \frac{-3 \pm \sqrt{9-370}}{20} = \frac{-3 \pm \sqrt{-361}}{20} = \frac{-3 \pm 19i}{20}$

The general solution is therefore $y(t) = e^{-3/20\, t}\left[C_1 \cos\left(\frac{19}{20}t\right) + C_2 \sin\left(\frac{19}{20}t\right)\right]$

where C_1, C_2 are constants which we fix using the initial conditions.

Notice that

$$y'(t) = -\frac{3}{20} e^{-\frac{3}{20}t}\left[C_1 \cos\left(\frac{19}{20}t\right) + C_2 \sin\left(\frac{19}{20}t\right)\right] + \frac{19}{20} e^{-\frac{3}{20}t}\left[-C_1 \sin\left(\frac{19}{20}t\right) + C_2 \cos\left(\frac{19}{20}t\right)\right]$$

Hence $\begin{cases} 0.1 = y(0) = C_1 \\ 0 = y'(0) = -\frac{3}{20}C_1 + \frac{19}{20}C_2 \end{cases}$, i.e. $\begin{cases} C_1 = 0.1 = \frac{1}{10} \\ C_2 = \frac{3}{19}C_1 = \frac{3}{190} \end{cases}$

The position of the mass at time t (in m) is

$$\boxed{y(t) = \frac{1}{10} e^{-3/20\, t}\left[\cos\left(\frac{19}{20}t\right) + \frac{3}{19}\sin\left(\frac{19}{20}t\right)\right]}$$

(d) Determine the quasi-period of the motion.

The quasi-frequence is $\nu = \frac{19}{20}$. So the quasi-period is

$$\boxed{T = \frac{2\pi}{\nu} = \frac{40}{19}\pi \sim 6.61 \text{ (in sec.)}}$$