PRACTICE 2 (ANSWERS; NO STEPS, NO JUSTIFICATIONS HERE)

↳ PLEASE DO NOT FORGET TO JUSTIFY DURING THE MIDTERM!!

EXERCISE 1: $x' = Ax$

(a) $A = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}$.

1. Eigenvalues/eigenvectors: $\lambda_1 = -1$, $v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$; $\lambda_2 = 2$, $v_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

   General solution: $x(t) = C_1 e^{-t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $t \in \mathbb{R}$; C_1, C_2 constants

2. $x(t) = -e^{-t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + e^{2t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $t \in \mathbb{R}$.

3. $(0,0)$ unique equilibrium point; a saddle; unstable.

4. If $C_2 \neq 0$, then $x(t) \underset{t \to +\infty}{\sim} C_2 e^{2t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ [trajectory moves toward ∞]

   If $C_2 = 0$, then $x(t) = C_1 e^{-t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ for $t \to +\infty$

   If $C_1 \neq 0$, then $x(t) \underset{t \to -\infty}{\sim} C_1 e^{-t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ [trajectory moves towards ∞]

   If $C_1 = 0$, then $x(t) = C_2 e^{2t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ for $t \to -\infty$

(b) $A = \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}$

1. Eigenvalues/eigenvectors: $\lambda_1 = 1$, $v_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$; $\lambda_2 = 4$, $v_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

   General solution: $x(t) = C_1 e^{t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + C_2 e^{4t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $t \in \mathbb{R}$; C_1, C_2 constants

2. $x(t) = e^{t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $t \in \mathbb{R}$.

3. $(0,0)$ unique equilibrium point; nodal source; unstable.

4. If $C_2 \neq 0$, then $x_2(t) \underset{t \to +\infty}{\sim} C_2 e^{4t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

   All trajectories move toward ∞ as $t \to +\infty$

   If $C_1 \neq 0$, then $x_2(t) \underset{t \to -\infty}{\sim} C_1 e^{t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

   All trajectories move backwards in time (i.e. for $t \to -\infty$) to $(0,0)$

(c) $A = \begin{pmatrix} -1/2 & 3 \\ 0 & 2 \end{pmatrix}$

1. Eigenvalues / eigenvectors: $\lambda_1 = -1/2$, $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$; $\lambda_2 = 2$; $\mathbf{v}_2 = \begin{pmatrix} 6 \\ 5 \end{pmatrix}$

   General solution: $\mathbf{x}(t) = C_1 e^{-t/2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 6 \\ 5 \end{pmatrix}$, $t \in \mathbb{R}$; C_1, C_2 constants

2. $\mathbf{x}(t) = \frac{11}{5} e^{-t/2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \frac{1}{5} e^{2t} \begin{pmatrix} 6 \\ 5 \end{pmatrix}$, $t \in \mathbb{R}$.

3. $(0,0)$ unique equilibrium point; a saddle; unstable.

4. If $C_2 \neq 0$, then $\mathbf{x}(t) \underset{t \to +\infty}{\sim} -C_2 e^{2t} \begin{pmatrix} 6 \\ 5 \end{pmatrix}$ [trajectory moves toward ∞]

   If $C_2 = 0$, then $\mathbf{x}(t) = C_1 e^{-t/2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ for $t \to +\infty$

   If $C_1 \neq 0$, then $\mathbf{x}(t) \underset{t \to -\infty}{\sim} C_1 e^{-t/2} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ [trajectory moves towards ∞]

   If $C_1 = 0$, then $\mathbf{x}(t) = C_2 e^{2t} \begin{pmatrix} 6 \\ 5 \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ for $t \to -\infty$

(d) $A = \begin{pmatrix} 0 & -1 \\ 2 & 2 \end{pmatrix}$

1. Eigenvalues / eigenvectors: $\lambda_1 = 1 + i$, $\mathbf{v}_1 = \begin{pmatrix} -1+i \\ 2 \end{pmatrix}$; $\lambda_2 = \bar{\lambda}_1$; $\mathbf{v}_2 = \bar{\mathbf{v}}_1$

   General solution: $\mathbf{x}(t) = e^t \left[C_1 \begin{pmatrix} -\cos t - \sin t \\ 2\cos t \end{pmatrix} + C_2 \begin{pmatrix} \cos t - \sin t \\ 2\sin t \end{pmatrix} \right]$

   $t \in \mathbb{R}$, C_1, C_2 constants

2. $\mathbf{x}(t) = \frac{1}{2} e^t \left[\begin{pmatrix} \cos t + \sin t \\ -2\cos t \end{pmatrix} + \begin{pmatrix} \cos t - \sin t \\ 2\sin t \end{pmatrix} \right]$, $t \in \mathbb{R}$

3. $(0,0)$ unique equilibrium point; a spiral source; unstable

4. $\lim_{t \to -\infty} \mathbf{x}(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$; the direction of the motion is away from the origin; the trajectories spiral and become unbounded

EXERCISE 2 (ANSWERS WITH DETAILS TO BE FILLED IN!)

(a) Standard form $y'' + \frac{3}{2t} y' - \frac{1}{2t^2} y = 0$ with continuous coefficients on $(-\infty, 0) \cup (0, +\infty)$. The longest interval containing $t=1$ on which their are continuous is $(0, +\infty)$

(b)

- $y_1(t) = t^{1/2}$ is a solution. Indeed, for $t>0$

$y_1(t) = t^{1/2}$, $y_1'(t) = \frac{1}{2} t^{-1/2}$, $y_1''(t) = -\frac{1}{4} t^{-3/2}$. Substituting into the DE one obtains:

$$2t^2 \cdot \left(-\frac{1}{4} t^{-3/2}\right) + 3t \cdot \frac{1}{2} t^{-1/2} - t^{1/2} = -\frac{1}{2} t^{1/2} + \frac{3}{2} t^{1/2} - t^{1/2} = 0$$

- $y_2(t) = t^{-1}$ is a solution. Indeed, for $t>0$,

$y_2(t) = t^{-1}$, $y_2'(t) = -t^{-2}$, $y_2''(t) = 2t^{-3}$. Hence, substituting into the DE one obtains:

$$2t^2 (2t^{-3}) + 3t(-t^{-2}) - t^{-1} = 4t^{-1} - 3t^{-1} - t^{-1} = 0$$

(c) $$W[y_1, y_2](t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} t^{1/2} & t^{-1} \\ \frac{1}{2} t^{-1/2} & -t^{-2} \end{vmatrix} = -t^{1/2 - 2} - \frac{1}{2} t^{-3/2} = -\frac{3}{2} t^{-3/2} \neq 0$$

for $t>0$

(d) $y(t) = C_1 y_1(t) + C_2 y_2(t) = C_1 t^{1/2} + C_2 t^{-1}$, C_1, C_2 constants [by (b) and (c)]

(e) Apply the method of variation of parameters. In standard form the given DE is: $y'' + \frac{3}{2}\frac{1}{t} y' - \frac{1}{2t^2} y = \frac{1}{2t}$. So $g(t) = \frac{1}{2t}$. A particular solution $Y(t)$ is:

$$Y(t) = -y_1(t) \int \frac{y_2(t) g(t)}{W[y_1, y_2](t)} dt + y_2(t) \int \frac{y_1(t) g(t)}{W[y_1, y_2](t)} dt$$

$$= \frac{2}{3} t^{1/2} \int \frac{1}{t} \frac{1}{2t} t^{3/2} \cdot dt - \frac{2}{3} t^{-1} \int t^{1/2} \cdot \frac{1}{2t} t^{3/2} dt$$

$$= \frac{1}{3}t^{1/2}\int t^{-1/2}\,dt - \frac{1}{3}t^{-1}\int t\,dt$$

$$= \frac{1}{3}t^{1/2}\left(\frac{1}{-\frac{1}{2}+1}t^{1/2}+C_1\right) - \frac{1}{3}t^{-1}\left(\frac{1}{2}t^2+C_2\right)$$

can choose C_1 e.g $=0$ because $t^{1/2}$ is solution of the homogeneous equation

can choose C_2 e.g. $=0$ because t^{-1} is solution of the homogeneous equation

$$= \frac{2}{3}t - \frac{1}{6}t = \frac{1}{2}t$$

(e) The general solution of $2t^2y''+3ty'-y=t$ is of the form

$$y(t) = y_c(t) + Y(t)$$

where y_c (= the complementary solution) is the general solution of the corresponding homogeneous DE and $Y(t)$ is a particular solution of the given DE

Thus: $y(t) = C_1t^{1/2} + C_2t^{-1} + \frac{1}{2}t$, C_1, C_2 constants, $t>0$

(f) Set $\begin{cases} x_1 = y \\ x_2 = y' \end{cases}$. Hence $\begin{cases} x_1' = y' = x_2 \\ x_2' = y'' = -\frac{3}{2t}y' + \frac{1}{2t^2}y + \frac{1}{2t} = \frac{1}{2t^2}x_1 - \frac{3}{2t}x_2 + \frac{1}{2t} \end{cases}$

i.e.

$$\begin{cases} x_1' = x_2 \\ x_2' = \frac{1}{2t^2}x_1 - \frac{3}{2t}x_2 + \frac{1}{2t} \end{cases}$$

In matrix form: $\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t)$, where

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix},\quad \mathbf{P}(t) = \begin{pmatrix} 0 & 1 \\ \frac{1}{2t^2} & -\frac{3}{2t} \end{pmatrix},\quad \mathbf{g}(t) = \begin{pmatrix} 0 \\ 1/2t \end{pmatrix}$$

(g) The general solution of the system in (f) is $\begin{cases} x_1(t) = y(t) \\ x_2(t) = y'(t) \end{cases}$

where $y(t) = C_1 t^{1/2} + C_2 t^{-1} + \frac{1}{2} t$ is the solution found in (10) and

Hence $y'(t) = \frac{C_1}{2} t^{-1/2} - C_2 \frac{1}{t^2} + \frac{1}{2}$

Thus $\mathbf{x}(t) = \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix} = C_1 \begin{pmatrix} t^{1/2} \\ \frac{1}{2} t^{-1/2} \end{pmatrix} + C_2 \begin{pmatrix} t^{-1} \\ -t^{-2} \end{pmatrix} + \frac{1}{2} \begin{pmatrix} t \\ 1 \end{pmatrix}$, $C_1, C_2 \in \mathbb{R}$, $t > 0$