Georgia Tech – Lorraine
Fall 2019
Differential Equations
Math 2552
10/10/2019

Last Name:
First Name:

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Quiz n^0 3 (20 minutes)

Show your work and justify your answers. Calculators, notes, cell phones, books are not allowed. Please do not use red or pink ink. Maximum: 20 points

Exercise 1 (2+3+2+3 points) .

Consider the system of differential equations
$$\begin{cases} \dfrac{dx}{dt} = -x+1 \\ \dfrac{dy}{dt} = -2y \end{cases}$$

(a) Suppose $x \neq 1$. Determine a first order differential equation for y as a function of x.

$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$. Hence, if $\frac{dx}{dt} \neq 0$, i.e. $x \neq 1$, we have $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2y}{x-1}$

(b) Solve the differential equation in (a) and determine a function $H(x, y)$ such that every solution satisfies an equation of the form $H(x, y) = C$, where C is a constant.
(Write $H(x, y)$ so that is does not contain any logarithmic terms.)

$\frac{dy}{dx} = \frac{2y}{x-1}$. If $y \neq 0$, then $\frac{1}{y}\frac{dy}{dx} = 2\frac{1}{x-1}$ (separable first order DE)

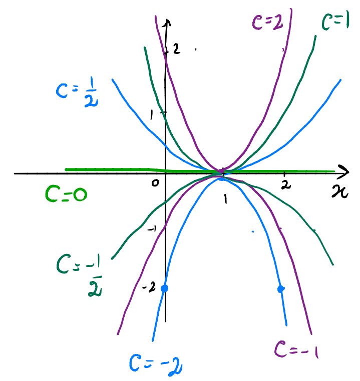
Integrate both sides wrt x and substitute $dy = \frac{dy}{dx}dx$: $\int \frac{1}{y}dy = 2\int \frac{1}{x-1}dx$

i.e. $\ln|y| = 2\ln|x-1| + C_1 = \ln((x-1)^2) + C_1$ with C_1 constant. Exponentiating, we obtain:

$|y| = e^{C_1}(x-1)^2$, i.e. $y = \pm e^{C_1}(x-1)^2$. So: $y = C(x-1)^2$ where C is an arbitrary constant ($C=0$ allows the solution $y=0$ too)

Thus $H(x,y) = \frac{y}{(x-1)^2}$

(c) Describe the level curves of the function $H(x, y)$ and sketch some of them.



$H(x,y) = C \iff y = C(x-1)^2$

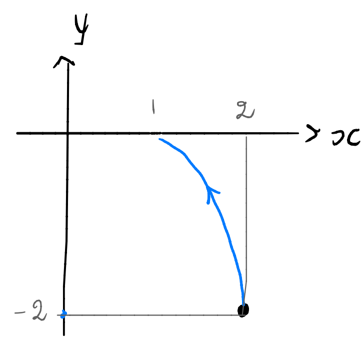
For $C \neq 0$, this is a family of parabolas with common vertex $(x_v, y_v) = (1,0)$

They are concave up for $C>0$ and concave down for $C<0$

They intersect the y-axis at $(0, C)$.

The level curve for $C=0$ is the line $y=0$ (the x-axis)

(d) For $t > 0$, sketch the trajectory corresponding to the initial condition $x(0) = 2$ and $y(0) = -2$ and indicate the direction of motion for increasing t.
(Sketch the trajectory only and not the level curve to which it belongs.)



$C = H(2,-2) = -2 \Rightarrow y = -2(x-1)^2$

As $t>0$ increases, the x value decreases to 1 and the y value increases to 0.

Please turn ⟶

Exercise 2 (2+4+2+2 points) .

A 1-kilogram mass stretches a spring 20 cm. The mass is pulled down 5 cm below its equilibrium position and given an initial upward velocity of 10 cm/s. Assume that there is no damping and recall that $g = 9.8\text{m/sec}^2$

(a) Determine the spring constant of this spring.

The spring and the gravitational force balance each others when the mass is at equilibrium, i.e. $mg - kL = 0$, where: $m = 1$ kg, $g = 9.8 m/sec^2$; L = extension of the spring from natural length to equilibrium position = 20 cm = 0.2 m, k = spring constant

Hence $k = \dfrac{1 \cdot 9.8}{0.2} = 49\ kg/sec^2 = 49\ N/m$

(b) Write an initial value problem (IVP) that models the motion of the mass.

(Choose a downward-pointing coordinate axis with origin at the equilibrium position. Do not solve this IVP)

Unforced, undamped oscillator $\Bigg\}$:

The equation of motion is:

$my'' + ky = 0$ i.e. $y'' + 49y = 0$

with initial conditions

$y(0) = 0.05$ m, $y'(0) = -0.1$ m/sec

(c) Introduce state variables and convert the IVP of (b) into an IVP for a system of two first-order linear differential equations. Use matrix notation.

(Do not solve this IVP)

State variables

$$x_1 = y, \quad x_2 = y' \Rightarrow \begin{cases} x_1' = x_2 \\ x_2' = -49\,x_1 \end{cases} \quad \text{with initial condition } \begin{cases} x_1(0) = 0{,}05 \\ x_2(0) = -0.1 \end{cases}$$

Matrix notation: $\mathbf{x}' = \underbrace{\begin{pmatrix} 0 & 1 \\ -49 & 0 \end{pmatrix}}_{A} \mathbf{x}$ with initial condition

$$\mathbf{x}(0) = \begin{pmatrix} 0{,}05 \\ -0.1 \end{pmatrix}$$

(d) Will the system oscillate indefinitely? Explain.

(A mathematical argument is expected.)

The characteristic equation of A is $\det(A - \lambda I) = 0$, i.e. $\lambda^2 + 49 = 0$, i.e. $\lambda = \pm 7i$

so A has two complex conjugate purely imaginary eigenvalues.

This yields a periodic solution.

Therefore the system will oscillate indefinitely.