Georgia Tech – Lorraine
Fall 2019
Differential Equations
Math 2552
10/24/2019

Last Name:
First Name:

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TOT	

Quiz n^0 4 (20 minutes)

Show your work and justify your answers. Calculators, notes, cell phones, books are not allowed. Please do not use red or pink ink. Maximum: 20 points

Exercise. [Each of the questions (a),(b),(c),(d) and (e) below is worth 4 points.]
An object of mass $m = 0.2$ kg is hung from a spring with spring constant $k = 40$ N/m. The object is subject to a damping with damping coefficient $\gamma = 4$ Ns/m.

(a) Suppose first that there is no external force acting on the spring-mass system.

Set up the differential equation of the motion.

(Choose the equilibrium point as the origin of a downward-pointing y-coordinate axis.)

The DE of the motion is: $my'' + \gamma y' + ky = 0$
i.e. $0.2\, y'' + 4y' + 40\, y = 0$
i.e. $y'' + 20y' + 200\, y = 0$

(b) Determine the general solution of the differential equation you found in part (a).

The characteristic equation $\lambda^2 + 20\lambda + 200 = 0$ has roots
$\lambda = -10 \pm \sqrt{100-200} = -10 \pm i10 = 10(-1 \pm i)$ (complex conjugate roots)
A fundamental system of real-valued solutions is $y_1(t) = e^{-10t}\cos(10t)$ and $y_2(t) = e^{-10t}\sin(10t)$. The general solution is
$y(t) = C_1 e^{-10t}\cos(10t) + C_2 e^{-10t}\sin(10t)$, where C_1, C_2 are arbitrary constants.

(c) Suppose that at time $t = 0$ the mass is pulled down 0.5 m below its equilibrium position and then released (i.e. the initial velocity is 0).

Determine the motion $y(t)$ of the mass as a function of the time t.

We determine the value of the constants C_1, C_2 in (1) using the initial conditions: $y(0) = 0.5\,m$, $y'(0) = 0\, m/s$.

We have $y(t) = e^{-10t}(C_1\cos(10t) + C_2\sin(10t))$.
Hence $y'(t) = -10e^{-10t}(C_1\cos(10t) + C_2\sin(10t)) + e^{-10t}(-10C_1\sin(10t) + 10C_2\cos(10t))$

So $\begin{cases} 0.5 = y(0) = C_1 \\ 0 = y'(0) = -10C_1 + 10C_2 \end{cases}$ i.e. $\begin{cases} C_1 = 0{,}5 \\ C_2 = C_1 = 0{,}5 \end{cases}$

Thus $y(t) = \frac{1}{2}e^{-10t}(\cos(10t) + \sin(10t))$.

(d) The following two questions consider the motion from a qualititative point of view:

(d1) Can the function $y(t)$ be written in the form $y(t) = h(t)\cos(\nu t - \delta)$ for a suitable function $h(t)$ and suitable constants ν and δ? If so, determine $h(t)$, ν and δ.

Recall that $\cos(\frac{\pi}{4}) = \sin(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$ and $\cos(\alpha-\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$, So

$$y(t) = \frac{1}{2}e^{-10t}(\cos(10t)+\sin(10t)) = \frac{\sqrt{2}}{2}e^{-10t}\left(\frac{\sqrt{2}}{2}\cos(10t) + \frac{\sqrt{2}}{2}\sin(10t)\right)$$

$= \frac{\sqrt{2}}{2}e^{-10t}\cos(10t - \frac{\pi}{4})$. Thus $y(t)$ can be written in the requested form with $h(t) = \frac{\sqrt{2}}{2}e^{-10t}$, $\nu = 10$, $\delta = \frac{\pi}{4}$.

(d2) What is the behavior of $y(t)$ as t increases?

$\lim\limits_{t\to+\infty} y(t) = 0$ because $\lim\limits_{t\to+\infty} e^{-10t} = 0$ and $t \to \cos(10t - \frac{\pi}{4})$ is bounded, with $|\cos(10t - \frac{\pi}{4})| \leq 1$ for all t.

(e) Suppose now that the mass-spring system is subject to a periodic force $F(t) = 100\sin(20t)$ N. Explain (*without computing it!*) how one can find a steady-state solution. Your explanation must include an initial guess for the particular solution.

The DE of the motion is now $y'' + 20y' + 200y = 100\sin(20t)$. It is a 2^{nd} order linear DE with nonhomogeneous term $g(t) = 100\sin(20t)$. The associated homogeneous equation $y'' + 20y' + 200y = 0$ has constant coefficients and it is not solved by the function $g(t)$. We can therefore find a steady-state solution of the given nonhomogenous DE by the method of undetermined coefficients with initial guess $Y_p(t) = A\cos(30t) + B\sin(30t)$. The constants A and B are determined by inserting $Y_p(t), Y_p'(t), Y_p''(t)$ into the DE and matching the coefficients of $\sin(30t)$ and $\cos(30t)$ in the resulting equation.

- Equivalently, we look of a particular solution of $y'' + 20y' + 200y = 100e^{20it}$ of the form $Y = Ce^{20it}$. We determine C by inserting $Y(t), Y'(t), Y''(t)$ in the DE, and finally $Y_p(t) = \operatorname{Im} Y(t)$ is a steady-state solution of the initial DE.