Georgia Tech – Lorraine
Fall 2019
Differential Equations
Math 2552
11/20/2019

Last Name:
First Name:

EX	
1	
2	
TOT	

Quiz n^0 5 (20 minutes)

Show your work and justify your answers. Calculators, notes, cell phones, books are not allowed. The table of Laplace transforms is allowed. Please do not use red or pink ink. Maximum: 20 points

Exercise 1 (4+6 points) .

(a) Write the following function using the unit step function

$$f(t) = \begin{cases} t & \text{if } 0 \le t < \pi \\ \cos(3t) & \text{if } \pi \le t < 2\pi \\ 0 & \text{if } t \ge 2\pi \end{cases}$$

$$f(t) = t\left(u_0(t) - u_\pi(t)\right) + \cos(3t)\left(u_\pi(t) - u_{2\pi}(t)\right)$$
$$= t + \left(\cos(3t) - t\right) u_\pi(t) - \cos(3t)\, u_{2\pi}(t) \qquad \text{for } t \geq 0$$

(b) Find the Laplace transform of the function f in (a).

$$f(t) = t + f_1(t-\pi)\, u_\pi(t) + f_2(t-2\pi)\, u_{2\pi}(t)$$

where

$f_1(t-\pi) = \cos(3t) - t$, i.e. $f_1(t) = \cos(3(t+\pi)) - t - \pi = -\cos(3t) - t - \pi$

because $\cos(3t + 3\pi) = \cos(3t + \pi) = -\cos(3t)$

$f_2(t-2\pi) = -\cos(3t)$, i.e. $f_2(t) = -\cos(3(t+2\pi)) = -\cos(3t)$

Thus for $s > 0$:

$$\mathcal{L}\{f\}(s) = \mathcal{L}\{t\}(s) + e^{-\pi s}\mathcal{L}\{f_1\}(s) + e^{-2\pi s}\mathcal{L}\{f_2\}(s)$$
$$= \mathcal{L}\{t\}(s) - e^{-\pi s}\mathcal{L}\{\cos(3t) + t + \pi\}(s) - e^{-2\pi s}\mathcal{L}\{\cos(3t)\}(s)$$
$$= \frac{1}{s^2} - e^{-\pi s}\left[\frac{s}{s^2+9} + \frac{1}{s^2} + \frac{\pi}{s}\right] - e^{-2\pi s}\frac{s}{s^2+9}$$
$$= \left(1 - e^{-\pi s}\right)\frac{1}{s^2} - \left(e^{-\pi s} + e^{-2\pi s}\right)\frac{s}{s^2+9} + e^{-\pi s}\cdot\frac{\pi}{s}$$

Please turn ⟶

Exercise 2 (10 points) . Solve the following initial value problem using Laplace transforms.

$$y'' + 2y' + 2y = e^{-t} \qquad \text{with initial conditions } y(0) = y'(0) = 0$$

Apply the Laplace transform to both sides of the DE and set $\mathcal{L}\{y\} = Y$

$$\mathcal{L}\{y''\}(s) + 2\,\mathcal{L}\{y'\}(s) + 2\,\mathcal{L}\{y\}(s) = \mathcal{L}\{e^{-t}\}(s)$$

$$[s^2 Y(s) - s\underbrace{y(0)}_{=0} - \underbrace{y'(0)}_{=0}] + 2[sY(s) - \underbrace{y(0)}_{=0}] + 2Y(s) = \frac{1}{s+1} \qquad \text{for } s > -1$$

$$(s^2 + 2s + 2)\,Y(s) = \frac{1}{s+1}. \text{ Thus } Y(s) = \frac{1}{s+1} \cdot \frac{1}{s^2+2s+2}$$

The polynomial s^2+2s+2 has no real roots, so we look for constants A, B, C such that

$$\frac{1}{(s+1)(s^2+2s+2)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+2s+2} = \frac{A(s^2+2s+2) + \overbrace{(s+1)(Bs+C)}^{Bs^2+(B+C)s+C}}{(s+1)(s^2+2s+2)} = \frac{(A+B)s^2 + (2A+B+C)s + (2A+C)}{(s+1)(s^2+2s+2)}$$

i.e. $(A+B)s^2 + (2A+B+C)s + (2A+C-1) = 0$ pour tous $s > -1$

i.e. $\begin{cases} A + B = 0 \\ 2A + B + C = 0 \\ 2A + C = 1 \end{cases} \Rightarrow \begin{array}{l} A = 1 \\ B = -1 \\ C = -1 \end{array}$. Thus $Y(s) = \frac{1}{s+1} - \frac{s+1}{s^2+2s+2}$

Completing the squares: $s^2 + 2s + 2 = (s+1)^2 + 1$

Hence

$$y(t) = \mathcal{L}^{-1}\{Y\}(t) = \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}(t) - \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+1}\right\}(t)$$

$$= e^{-t} - \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\Big|_{s \to s+1}\right\}(t)$$

$$= e^{-t} + e^{-t}\cos t$$

$$= e^{-t}(1 + \cos t)$$